Principles of Digital Communications
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FINAL

You have 3 hours. It is not necessarily expected that you finish all problems. Do not loose too much time on each problem but try to collect as many points as possible.

Closed-book, no calculators, cell-phones, crypt sheets or stolen glances to your neighbors are allowed. Write only what is relevant to the question!

Good Luck!!

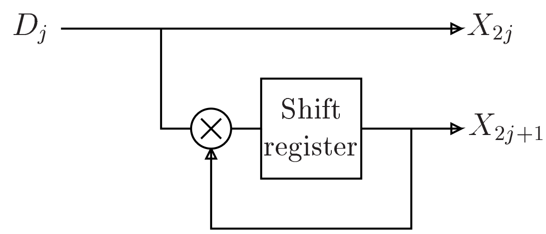
Name: ______________________________

Prob I	/ 30
Prob II	/ 15
Prob III	/ 15
Prob IV	/ 20
Prob V	/ 20
Total	/ 100

I. **Convolution Code and Viterbi Algorithm [30 points]**

a. Convolutional Encoder

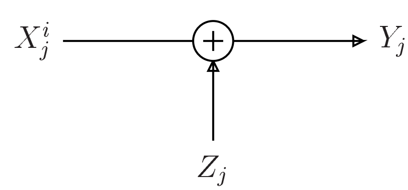
Consider the following convolutional encoder. The input sequence D_j takes values in $\{-1, +1\}$ for $j = 0, 1, 2, \cdots, k-1$. The output sequence, call it X_j, $j = 0, \cdots, 2k-1$, is the result of passing D_j through the filter shown below, where we assume that the initial content of the memory is 1.



1. In the case $k = 3$, how many different hypotheses can the transmitter send using the input sequence (D_0, D_1, D_2), call this number m.
1. Draw the finite state machine corresponding to this encoder. Label the transitions with the corresponding input and output bits. How many states does this finite state machine have?
2. Draw a section of the trellis representing this encoder.
3. What is the rate of this encoder?
(number of information bits /number of transmitted bits).

b. Viterbi Decoder

Consider the channel defined by $Y_j = X_j^i + Z_j$. Let X_j^i denote the ouput of the convolutional encoder at time j when we transmit hypothesis i, $i = 0, \cdots, m-1$. Further, assume that Z_j is a zero-mean Gaussian random variable with variance $\sigma^2 = 4$ and let Y_j be the output of the channel.



Assume that the received vector is $\bar{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (1, 2, -2, -1, 0, 3)$. It is the task of the receiver to decide which hypothesis i was chosen or, equivalently, which vector $\bar{X}^i = (X_1^i, X_2^i, X_3^i, X_4^i, X_5^i, X_6^i)$ was transmitted.

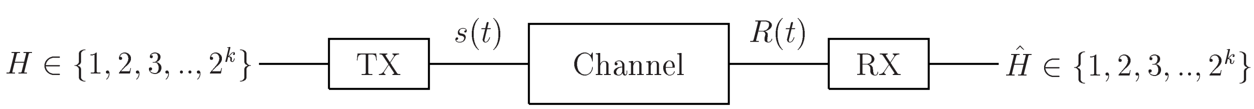
1. Without using the Viterbi algorithm, write formally (in terms of $\bar{Y}$ and $\bar{X}^i$) the optimal decision rule. Can you simplify this rule to express it as a function of inner products of vectors? In that case, how many inner products do you have to compute to find the optimal decision?
2. Use the Viterbi algorithm to find the most probable transmitted vector $\bar{X}^i$.

c. Performance Analysis.

1. Draw the detour flow graph corresponding to this decoder and label the edges by the input weight using the symbol I, the output weight (of both branches) using the symbol D.

II. **Communication Chain [15 points]**

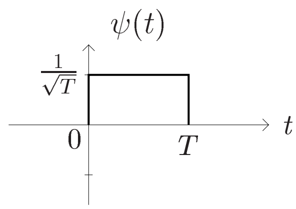
Consider the following communication chain, where we have 2^k possible hypotheses with $k \in \mathbb{N}$. TX represents the transmitter and RX the receiver and $s(t)$ and $R(t)$ represent respectively the transmitted and received waveforms.



a. Transmitter: "Bit by bit on a pulse train" signaling.

1. For the "bit by bit on a pulse train" signaling scheme, what is the dimension of the space spanned by the transmitted waveforms as a function of $k \in \mathbb{N}$?
2. Assume we use two symmetric real numbers $\{+a, -a\}$ as our antipodal values. What is the value of a that gives us signals of energy equal to one as a function of $k \in \mathbb{N}$?

3. You are given the waveform $\psi(t)$ depicted below



We want to transmit the hypothesis associated to the sequence $(a, -a, a, -a, -a)$. Plot the waveform you will send on the channel if you use the "bit by bit on a pulse train" signaling scheme.

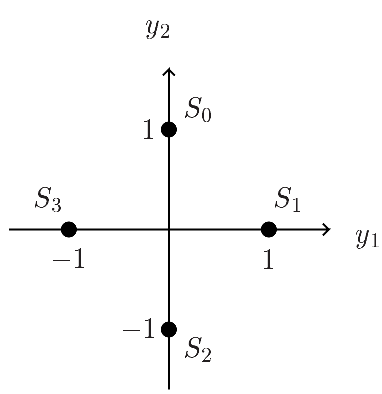
b. Receiver.

Let the received signal be $R(t) = s_i(t) + N(t)$ when the waveform $s_i(t)$ is transmitted. $N(t)$ denotes a white Gaussian noise process with double-sided power spectral density $\frac{N_0}{2}$.

1. Sketch the optimal receiver.
   What is the minimum number of filters you need for the optimal receiver? Explain.

III. **Decision Problem [15 points]**

Consider the following decision problem. For the hypothesis $H = i$, $i \in \{0, 1, 2, 3\}$, we send the point S_i, as shown in the figure below. The receiver observes the vector $Y = S_i + Z$, with Z a zero-mean Gaussian random vector whose covariance matrix is $\Sigma_Z = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.



1. Plot the optimal decision regions.

2. Before taking the optimal decision, can you think of a reversible transformation you can apply to the received vector Y in order to simplify the decision problem? In that case, draw the resulting transmitted points and the decision regions associated to them.
3. What is the error probability in this decision problem?

IV. **Equivalent Baseband Signal** [**20 points**]

1. You are given a "passband" signal $\psi(t)$ whose spectrum is centered around f_0. Write down in a generic form the different steps needed to find the baseband equivalent signal.
2. Consider the waveform
$$\psi(t) = \text{sinc}\left(\frac{t}{T}\right)\cos(2\pi f_0 t).$$
What is the equivalent baseband signal of this waveform.
3. Assume that the signal $\psi(t)$ is passed through the filter with impluse response $h(t)$ where $h(t)$ is specified by its baseband equivalent impulse response $h_E(t) = \frac{1}{T\sqrt{2}}\text{sinc}^2\left(\frac{t}{2T}\right)$. What is the output signal, both in passband as well as in baseband?

Hint: The Fourier transform of $\cos(2\pi f_0 t)$ *is* $\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$. *The Fourier transform of* $\frac{1}{T}sinc(\frac{t}{T})$ *is equal to* $\mathbf{1}_{[-\frac{1}{2T},\frac{1}{2T}]}(f)$ *with* $\mathbf{1}_{[-\frac{1}{2T},\frac{1}{2T}]}(f) = 1$ *if* $f \in [-\frac{1}{2T},\frac{1}{2T}]$ *and* 0 *otherwise.*

V. **A Simple Multiple-Access Scheme** [**20 points**]

Consider the following very simple model of a multiple-access scheme. There are two users. Each user has two hypotheses. Let $\mathcal{H}^1 = \mathcal{H}^2 = \{0,1\}$ denote the respective set of hypotheses and assume that both users employ a uniform prior. Further, let X^1 and X^2 be the respective signals sent by user one and two. Assume that the transmissions of both users are independent and that $X^1 \in \{\pm 1\}$ and $X^2 \in \{\pm 2\}$ where X^1 and X^2 are positive if their respective hypothesis is zero and negative otherwise. Assume that the receiver observes the signal $Y = X^1+X^2+Z$, where Z is a zero mean Gaussian random variable with variance σ^2 and is independent of the transmitted signal.

(a) Assume that the receiver observes Y and wants to estimate *both* transmitted signals, i.e., the receiver forms the estimate $\hat{H} = (\hat{H}^1, \hat{H}^2)$. Starting from first principles, what is the generic form of the optimal decision rule?

(b) For the specific set of signals given, what is the set of possible observations assuming that $\sigma^2 = 0$? Label these signals by the corresponding (joint) hypotheses.

(c) Assuming now that $\sigma^2 > 0$, draw the optimal decision regions.

(d) What is the resulting probability of correct decision? i.e., determine the probability $P\{\hat{H}^1 = H^1, \hat{H}^2 = H^2\}$.

(e) Finally, assume that we are only interested in the transmission of user two. What is $P\{\hat{H}^2 = H^2\}$?