Chapter 5

Communication Across Lowpass AWGN channels

5.1 Introduction

The previous chapters dealt with any communication system for the continuous-time additive white Gaussian noise channel.

In most situations of practical interest, there is a fairly strict bandwidth constraint on the transmitted signals. This is the case, for instance, if we communicate through the telephone line. Since the telephone line was designed to carry voice, there are filters that essentially block all frequency components above 4 KHz[1].

In this Chapter, we assume that the transmitted signal has to be a low-pass signal, i.e., a signals $s(t)$ such that $s_{\mathcal{F}}(f)$ vanishes for $f \geq B$ for some bandwidth B. The more general bandpass case where $s_{\mathcal{F}}(f)$ has to vanish for $|f| \notin [f_0 - \frac{B}{2}, f_0 + \frac{B}{2}]$ for some center frequency f_0 and bandwidth B is an extension that can be dealt with by means of an additional level of signal processing[2]. This additional signal processing is represented by the top block of Figure 1.2 and dealt with in details in the next chapter.

For the sake of designing a communication system, it does not matter whether the bandwidth constraint comes from the channel, like for the telephone line, or from frequency regulations (this would be the case in wireless communication). To remind ourselves about the presence of the bandwidth constraint, we insert a filter in our channel model as shown in Figure 5.1.

For now we assume that $h(t)$ is the channel impulse response of an ideal lowpass filter,

[1]The signal of a phone modem goes through such a filters. The filter can be bypassed and this is done if you request to install an ADSL modem.

[2]Wireless communication requires that we know how to handle the bandpass case. See e.g. http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html and click on "FM radio" for the spectrum used by FM radio stations or click on "electromagnetic spectrum" for a broader perspective

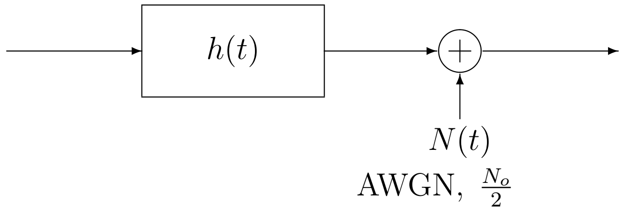


Figure 5.1: Lowpass channel model.

i.e.,

$$h_{\mathcal{F}}(f) = \begin{cases} 1, & |f| \leq B \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately, the presence of a filter in the channel model does not change or general approach. We have used the fundamental fact that the normed space spanned by the signaling waveforms used at the transmitter may be be represented by n-tuples. This has allowed us to define an interface (represented by the middle layer in Figure 1.2), that transforms a continuous-time AWGN channel into the n-tuple channel studied in Chapter 2.

The same approach will work here, but instead of fixing the signaling waveforms and finding a basis for the space that they span, we start with a basis that spans the subspace of $\mathcal{L}_2$ that fulfills the bandwidth constraint.

We will see that the sampling theorem is just a special kind of orthonormal expansion and that the basis it uses is the one that we need (at least in principle). Luckily, the orthonormal basis used in the sampling theorem has the convenient form $\psi_j(t) = \psi(t-jT)$ assumed in the previous chapter. Recall that we like this form since it allows us to obtain the projection $y_j = \langle R(t), \psi_j(t) \rangle$ by means of a single filter. Namely, $y_j = \langle R(t), \psi(t-jT) \rangle$ is output at time jT of the filter with input $R(t)$ and impulse response $\psi^*(-t)$. In this chapter, we will also derive the frequency-domain characterization of all orthonormal bases that have the convenient form $\psi_j(t) = \psi(t-jT)$ and use this result to pick orthonormal bases that are more practical than the $\psi(t) = \operatorname{sinc}(t/T)$ used in the sampling theorem.

5.2 Using The Sampling Theorem

THEOREM 45. *(Sampling Theorem)*[3] *Let $s(t)$ be a function in $\mathcal{L}_2$ that is lowpass limited to B. Then $s(t)$ is specified by its values at a sequence of points spaced $T = \frac{1}{2B}$ apart. In particular*

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT) \; sinc(\frac{t}{T} - n) \tag{5.1}$$

[3]See the Appendix for a proof of the sampling Theorem.

where $sinc(t) = \frac{\sin(\pi t)}{\pi t}$. $\square$

The sinc pulse does not have unit energy. Hence we define (its normalized version) $\psi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}(\frac{t}{T})$. The set $\{\psi(t - jT)\}_{j=-\infty}^{\infty}$ forms an orthonormal set. Hence (5.1) can be rewritten as

$$s(t) = \sum_{j=-\infty}^{\infty} s_j \psi(t - iT), \qquad \psi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}(\frac{t}{T}) \tag{5.2}$$

where $s_i = s(nT)\sqrt{T}$. This highlights the way the sampling theorem should be seen, namely as a particular instance of an orthonormal expansion. In this expansion the basis is formed by time translates of sinc pulses. Implicit in the sampling theorem is that the set $\{\psi(t - iT)\}_{i=-\infty}^{\infty}$ is a complete orthonormal basis for the set of waveforms that are lowpass limited to $B = \frac{1}{2T}$.

Now let us go back to our communication problem. The channel filter is a lowpass. Hence, any component of the transmitted signal $s(t)$ that lies outside the frequency range $[-B, B]$ will not be visible to the receiver; we may as well limit ourselves to signals that vanish at frequencies larger than B. All such signals have the form (5.2). Hence, without loss of generality, we may decide to transmit only signals of the form (5.2). For these signals the filter is transparent, which implies that the optimal receiver derived so far for the AWGN channel (without lowpass filter) is also optimal for the lowpass channel at hand.

Figure 5.2 shows the system from the encoder output (or from the source output if there is no encoder) to the receiver front end. If there is an encoder, the outputs of the receiver front end can be passed to a Viterbi algorithm to find the ML sequence estimate in an efficient way as described in the previous chapter. If there is no encoder, i.e., we are doing symbol-by-symbol on a pulse train possibly without restricting the alphabet of s_j to be binary, a ML decoder may decide on a symbol by symbol basis (i.e. it decides about s_j by looking at Y_j only).

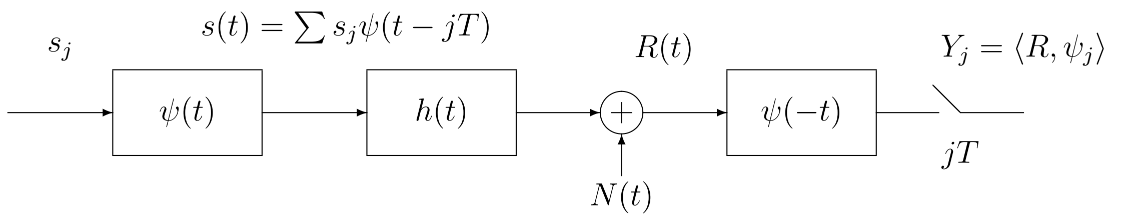


Figure 5.2: Lowpass system.

The details of what we may call the vector channel are shown in Figure 5.2. From the input/output point of view an equivalent channel model is the discrete-time AWGN channel shown in the next figure.

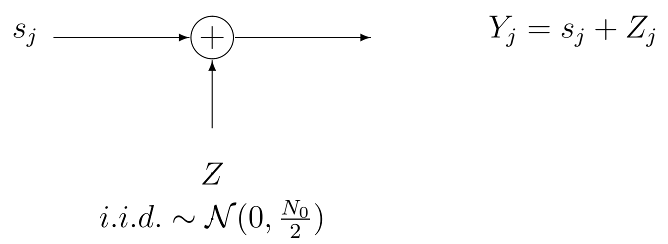


Figure 5.3: Equivalent discrete time channel.

This is the time to remind ourselves that the (vector) AWGN channel considered in Pass I was indeed fundamental.

Using the sinc function to modulate data has problems in practice. The most serious problem is that the sinc function drops off very slowly in time. One consequence of this is that a small error is the sampling time at the output of the matched filter results in serious *intersymbol interference.*

For a fixed bandwidth B, the sinc function allows us to send one symbol every T seconds through the equivalent discrete-time channel of Figure5.2. In the next section we will see that, if we are willing to use the equivalent discrete-time channel at a slightly lower rate, then a number of more practical solutions become available.

5.3 Using Nyquist Pulses

In the previous section we have constructed our orthonormal basis via $\psi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t}{T}\right)$. We wonder if there are other functions $\psi(t)$ that can be used instead of the sinc. We are looking for functions $\psi(t)$ with the property

$$\int_{-\infty}^{\infty} \psi(t - nT)\psi^*(t)dt = \delta_n. \tag{5.3}$$

Another example of such a function is $\psi(t) = 1_{[0,T]}(t)$, but this function is not bandlimited. Our aim is to find an insightful frequency-domain equivalent of (5.3). We now derive such a characterization which is known as Nyquist criterion. First define the following periodic function of period $\frac{1}{T}$

$$g(f) = \sum_{k \in \mathbb{N}} \psi_{\mathcal{F}}(f + \frac{k}{T})\psi^*_{\mathcal{F}}(f + \frac{k}{T}).$$

We can now transform (5.3) as follows:

$$\begin{aligned}\delta_n &= \int_{-\infty}^{\infty} \psi(t-nT)\psi_{\mathcal{F}}^*(t)dt \\ &\overset{(a)}{=} \int_{-\infty}^{\infty} \psi_{\mathcal{F}}(f)\psi_{\mathcal{F}}^*(f)e^{-j2\pi nTf}df \\ &\overset{(b)}{=} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} g(f)e^{-j2\pi nTf}df, \end{aligned} \tag{5.4}$$

where in (a) we used Parseval's relationship and the shift property of the Fourier transform, and in (b) we made repeated use of the fact that for an arbitrary function $u(x)$ and an arbitrary interval $[-\frac{a}{2}+ia, \frac{a}{2}+ia]$ in the domain of u,

$$\int_{-\frac{a}{2}+ia}^{\frac{a}{2}+ia} u(x)dx = \int_{-\frac{a}{2}}^{\frac{a}{2}} u(x+ia)dx.$$

But (5.4) is $2B$ (or equivalently $\frac{1}{T}$) times the n-th Fourier series coefficient of $g(f)$. Hence $f(f)$ is a periodic function with vanishing Fourier coefficients except for the coefficient with $n=0$. This means that the function $g(f)$ must be constant. Specifically, $g(f) \equiv T, f \in [-\frac{1}{2T}, \frac{1}{2T}]$.

We have proved the following

THEOREM 46. *(Nyquist). A waveform $\psi(t)$ is orthonormal to each shift $\psi(t-nT)$ if and only if*

$$\sum_{k=-\infty}^{\infty} |\psi_{\mathcal{F}}(f+\frac{k}{T})|^2 = T \quad \textit{for } f \in [-\frac{1}{2T}, \frac{1}{2T}] \tag{5.5}$$

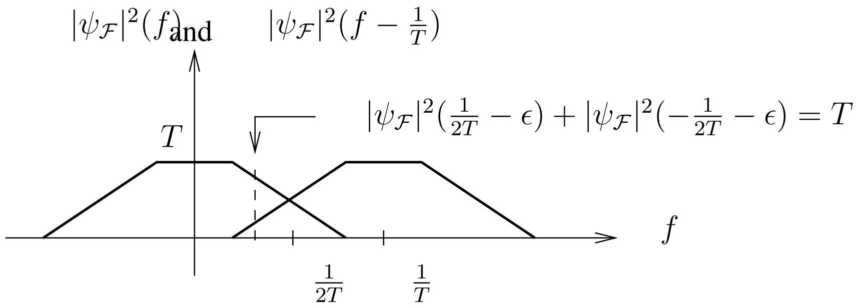
□

A few comments are in order:

- The sinc pulse is just a special case of a Nyquist pulse. It has the smallest possible bandwidth, namely $1/2T$ [Hz], among all pulses that satisfy Nyquist criterion for a given T. (Draw a picture if this is not clear to you).
- Normally we are interested in Nyquist pulses that have small bandwidth, typically between $1/2T$ and $1/T$. For pulses that are strictly bandlimited to $1/T$ or less, the Nyquist criterion is satisfied if and only if $|\psi_{\mathcal{F}}|^2(\frac{1}{2T}-\epsilon) + |\psi_{\mathcal{F}}|^2(-\frac{1}{2T}-\epsilon) = T$ for $\epsilon \in [\frac{1}{2T}, \frac{1}{2T}]$ (See picture below). If we assume (as we do) that $\psi(t)$ is real-valued, then $|\psi_{\mathcal{F}}|^2(-f) = |\psi_{\mathcal{F}}|^2(f)$. In this case the above relationship is equivalent to

  $$|\psi_{\mathcal{F}}|^2(\frac{1}{2T}-\epsilon) + |\psi_{\mathcal{F}}|^2(\frac{1}{2T}+\epsilon) = T, \qquad \epsilon \in [0, \frac{1}{2T}].$$

  This means that $|\psi_{\mathcal{F}}|^2(\frac{1}{2T}) = \frac{T}{2}$ and the amount by which $|\psi_{\mathcal{F}}|^2(f)$ increases when we go from $f = \frac{1}{2T}$ to $f = \frac{1}{2T} - \epsilon$ equals the decrease when we go from $f = \frac{1}{2T}$ to $f = \frac{1}{2T} + \epsilon$.



- Even though the Nyquist criterion is (5.5), to see if a pulse $\psi(t)$ fulfills Nyquist criterion we can either check if the pulse fulfills (5.3) or, alternatively, check if its Fourier transform $\psi_{\mathcal{F}}(t)$ fulfills (5.5). More often than not we are interested in pulses that are bandlimited. In those cases it is often easier to do the Fourier-domain check (5.5). However, if you are given a rectangular time-domain pulse, the obvious thing to do is to see whether or not it passes the time-domain test (5.3).

- If we use a Nyquist pulse $\psi(t)$ to generate

$$s(t)=\sum_{i=-\infty}^{\infty} s_i\psi(t-iT)$$

  Then the system depicted in Figure 5.2 is optimal.

- Verify that any pulse $\psi(t)$ such that

$$|\psi_{\mathcal{F}}|^2(f)=\begin{cases} T(1-T|f|), & |f|\le\frac{1}{T}\\ 0, & \text{otherwise}\end{cases}$$

  fulfills Nyquist criterion.

- Verify that any pulse $\psi(t)$ that satisfies

$$|\psi_{\mathcal{F}}|^2(f)=\begin{cases} T, & |f|\le\frac{1-\alpha}{2T}\\ \frac{T}{2}\left(1+\cos\left[\frac{\pi T}{\alpha}\left(|f|-\frac{1-\alpha}{2T}\right)\right]\right), & \frac{1-\alpha}{2T}<|f|<\frac{1+\alpha}{2T}\\ 0, & \text{otherwise}\end{cases}$$

  for some $\alpha\in(0,1)$, fulfills Nyquist criterion. Such a pulse is called a root-raised-cosine pulse. (Draw a picture).

5.4 Appendix: Fourier Series and Sampling Theorem

We briefly review the Fourier series focusing on the big picture and on how to remember things.

Let $f(x)$ be a periodic function, $x \in \mathbb{R}$. It has period p if $f(x) = f(x+p)$ for all $X \in \mathbb{R}$. Its fundamental period is the smallest such p. We are using the "physically unbiased" variable x instead of t (which usually represents time) since we want to emphasize that we are dealing with a general periodic function, not necessarily a function of time.

The periodic function $f(x)$ can be represented as a linear combination of complex exponentials of the form $e^{j2\pi \frac{x}{p} i}$. These are all the complex exponentials that have period p. Hence

$$f(x) = \sum_{i \in \mathbb{Z}} A_i \, e^{j2\pi \frac{x}{p} i}$$

for some sequence of coefficients $\ldots A_{-1}, A_0, A_1, \ldots$ Hence a function of fundamental period p may be written as a linear combination of all the complex exponentials of period p. This may be easily remembered.

The expression for A_i can also be easily remembered (derived). Two functions of fundamental period p are identical iff they coincide over a period. Hence it is sufficient to show that

$$f(x) 1_{[-\frac{p}{2}, \frac{p}{2}]}(x) = \sum_{i \in \mathbb{Z}} \sqrt{p} A_i \frac{e^{j\frac{2\pi}{p} xi}}{\sqrt{p}} 1_{[-\frac{p}{2}, \frac{p}{2}]}(x).$$

Since $\phi_i(x) = \frac{e^{j\frac{2\pi}{p} xi}}{\sqrt{p}} 1_{[-\frac{p}{2}, \frac{p}{2}]}(x)$, $i \in \mathbb{Z}$, is an orthonormal basis, the right side of the alone expression is one orthonormal expansion of the left. The coefficients of an orthonormal expansion are always found in the same way, namely

$$\sqrt{p} A_i = \langle f, \phi \rangle.$$

Hence

$$A_i = \frac{1}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x) e^{-j\frac{2\pi}{p} xi} dx$$

We hope that this will make it easier for you to remember (or re-derive) the formulas relating a periodic function and its Fourier series coefficients.

As an example of the utility of this relationship we derive the sampling theorem. Recall that the sampling theorem states that any $\mathcal{L}_2$ function $s(t)$ which is bandlimited to B may be written as

$$s(t) = \sum_k s(kT) \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

where $T = \frac{1}{2B}$.

Proof of the sampling theorem: By assumption, $s_{\mathcal{F}}(f) = 0$, $f \notin [-B, B]$. Hence,

$$s_{\mathcal{F}} = \sum_k A_i e^{+j\frac{2\pi}{2B} fk} 1_{[-B,B]}(f).$$

Taking to Fourier transform on both sides, using

$$s_{[-B,B]}(f) \Leftrightarrow \frac{1}{T} \operatorname{sinc}(\frac{t}{T}), \quad T = \frac{1}{2B}$$

and the time shifting property

$$h(t-\tau) \Leftrightarrow h_{\mathcal{F}}(f)e^{-j2\pi f\tau}$$

we obtain

$$s(t) = \sum \frac{A_k}{T} \operatorname{sinc}\left(\frac{t+kT}{T}\right).$$

We still need to determine $\frac{A_k}{T}$. It is straightforward to determine A_l from its definition, but it is easier to observe that if we plug in $t = nT$ on both sides of the expression above we obtain $s(nT) = \frac{A_{-n}}{T}$. This completes the proof. Since it is straightforward, we also determine A_l from the definition:

$$A_l = \frac{1}{2B}\int_{-B}^{B} s_{\mathcal{F}}(f)e^{-j\frac{2\pi}{2B}lf}df = \frac{1}{2B}\int_{-\infty}^{\infty} s_{\mathcal{F}}(f)e^{-j\frac{2\pi}{2B}lf}df = Ts(-lT),$$

where the first equality is the definition of the Fourier coefficient A_l, the second uses the fact that $s_{\mathcal{F}}(f) = 0$ for $f \notin [-B, B]$, and the third is the inverse Fourier transform evaluated at $t = -lT$. □

5.5 Problems

PROBLEM 1. *Consider the transmitted signal,* $S(t) = \sum_i X_i \psi(t - iT)$, *where* $X_i \in \{\pm 1\}$ *are i.i.d random variables and* $\{\psi(t-iT)\}_{i=-\infty}^{\infty}$ *forms an orthonormal set. Let* $Y(t)$ *be the matched filter output at the receiver. Then in the absence of noise,* X_i*'s are the samples of* $Y(t)$, *sampled at integer multiples of* T *i.e* $Y(iT) = X_i$. *In this MATLAB exercise we will try to see how crucial it is to sample at* $t = iT$ *as opposed to* $t = iT + \epsilon$. *Towards that goal we plot the so-called eye diagram.*

An eye diagram is the plot of $Y(t + iT)$ *versus* $t \in [-\frac{T}{2}, \frac{T}{2}]$, *plotted on top of each other for each* $i = 0 \cdots K-1$, *where* K *is the number of transmitted symbols. Thus at* $t = 0$ *on the eye diagram lies our sampling points mentioned earlier.*
(a) *Assuming* $K = 100$, $T = 1$ *and 10 samples per time period* T, *plot the eye diagrams when,*

(i) $\psi(t)$ *is a raised cosine with* $\alpha = 1$.

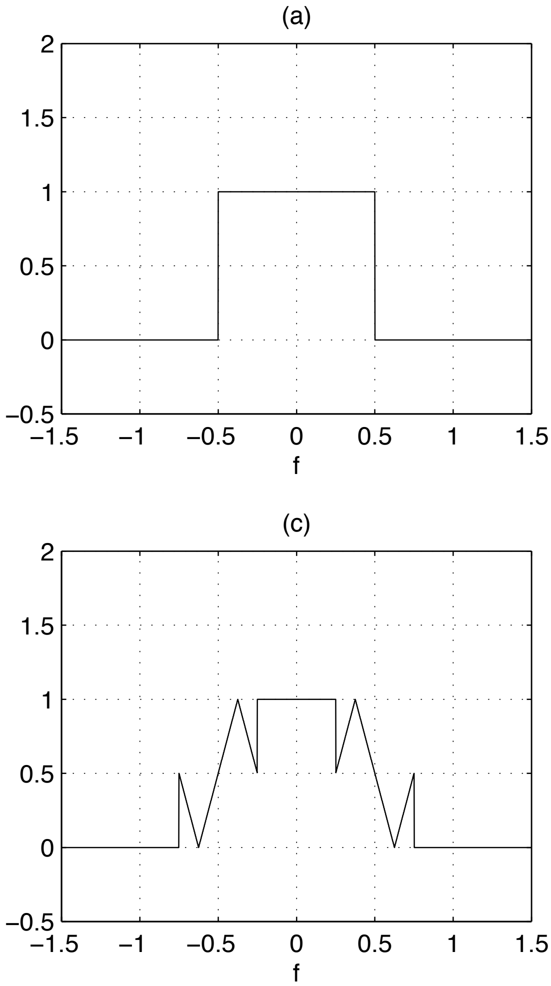
(ii) $\psi(t)$ *is a raised cosine with* $\alpha = \frac{1}{2}$.

(iii) $\psi(t)$ *is a raised cosine with* $\alpha = 0$ *(or sinc).*

(b) *From the plotted eye diagrams what can you say about the cruciality of the sampling points with respect to* α.

PROBLEM 2. (Nyquist Pulses.)

(i) *Consider the following* $|\theta_{\mathcal{F}}(f)|^2$. *The unit on the frequency axis is* $1/T$ *and the unit on the vertical axis is* T. *Which ones correspond to Nyquist pulses* $\theta(t)$ *for symbol rate* $1/T$? Note: *Figure* (d) *shows a* sinc^2 *function.*

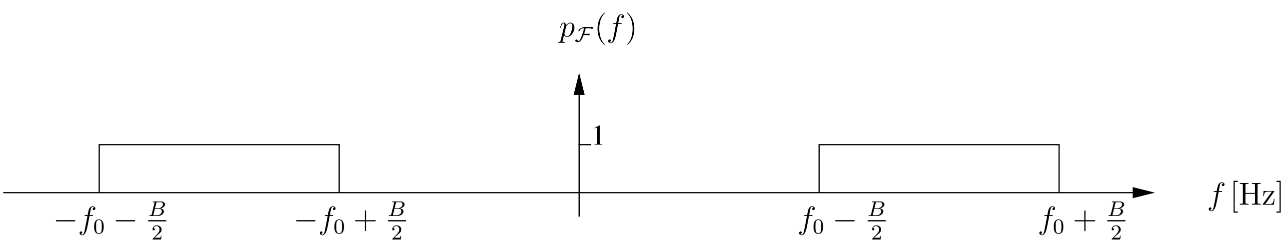


(ii) *Design a (non-trivial) Nyquist pulse yourself.*

(iii) *Sketch the block diagram of a binary communication system that employs Nyquist pulses. Write out the formula for the signal after the matched filter. Explain the advantages of using Nyquist pulses.*

Problem 3. (Nyquist Pulses.)

Consider a pulse $p(t)$ *defined via its Fourier transform* $p_{\mathcal{F}}(f)$ *as follows:*



(a) *What is the expression for* $p(t)$ *? (If you can't determine a mathematical expression, you may draw* $p(t)$ *qualitatively).*

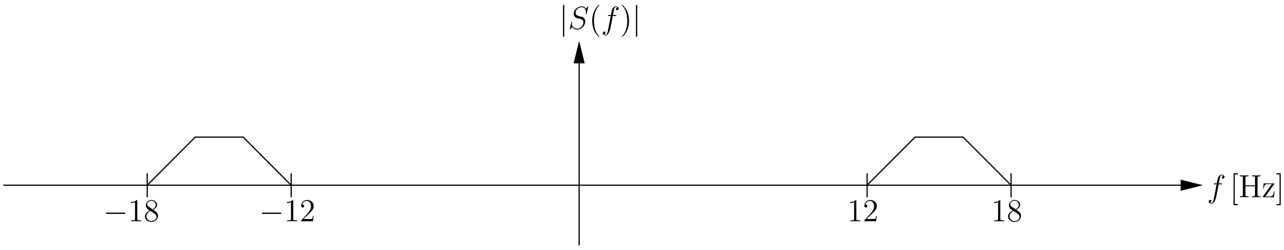
(b) *Determine the constant* c *so that* $\psi(t) = cp(t)$ *has unit energy.*

(c) *Assume that* $f_0 - \frac{B}{2} = B$ *and consider the infinite set of functions* $\cdots,\ \psi(t+T)$, $\psi(t)$, $\psi(t-T)$, $\psi(t-2T)$, $\cdots$. *Do they form an orthonormal set for* $T = \frac{1}{2B}$ *? (Explain).*

(d) *Determine all possible values of* $f_0 - \frac{B}{2}$ *so that* $\cdots,\ \psi(t+T),\ \psi(t),\ \psi(t-T)$, $\psi(t-2T),\ \cdots$ *forms an orthonormal set.*

PROBLEM 4. (Bandpass Sampling)

Consider the signal $s(t)$ *whose Fourier transform* $S(f)$ *has the property that* $|S(f)| = 0$ *for* $|f| \leq 12\,Hz$ *and for* $|f| \geq 18\,Hz$*, as illustrated in the following figure:*



For the following sampling frequencies f_s, *indicate (with yes or no) whether or not the signal* $s(t)$ *can be reconstructed from its samples taken every* $T_s = \frac{1}{f_s}$ *seconds apart.*

(i) $f_s = 10\,Hz$

(ii) $f_s = 12\,Hz$

(iii) $f_s = 14\,Hz$

(iv) $f_s = 16\,Hz$

(v) $f_s = 18\,Hz$

Communications Across Bandpass AWGN Channels

In the last part of this course we consider communication across bandpass AWGN channels. The block diagram of a general channel model is shown in Fig. 5.4. It looks much as the lowpass channel model considered in the previous chapter, but the filter's frequency response is now that of an *ideal bandpass filter*, i.e.,

$$h_{\mathcal{F}}(f) = \begin{cases} 1, & ||f| - f_0| \leq B \\ 0, & \text{otherwise.} \end{cases}$$

As before $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$.

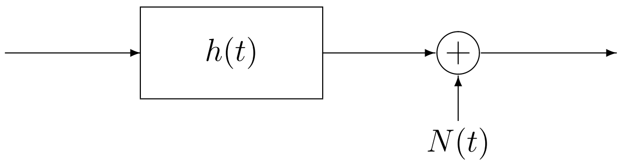


Figure 5.4: Bandpass AWGN Channel

There are various reasons for being interested in knowing how to communicate across a bandpass AWGN channel. Some of them are rooted into physics and some are dictated by practical choices. Among the former we mention that in wireless communications the channel seen between the transmit and the receiver antenna is always a bandpass channel (low frequency components do not generate electromagnetic waves capable of traveling long distances with small attenuation). The usable bandwidth of this channel is typically quite large however (it depends on the antennas among other things). More severe restrictions are dictated by international agreements that specify which portion of the electromagnetic spectrum can be used for what.

Regardless whether we have decided to use the better portion of the bandpass determined by physical constraints, or because we are complying with international regulations, what we "see" is often a bandpass channel as the one in Fig. 5.4.

If f_0 and B are in a certain relationship (can you tell which? …try with a picture), then $\psi(t) = h(t)/ \parallel h \parallel$ fulfills Nyquist criterion. (In the derivation of Nyquist criterion there is nothing that requires the pulse to be low-pass). In this case we can proceed exactly as in the previous chapter using this $\psi(t)$ as the basic pulse.

Now we proceed to derive an alternative (more general) approach that works regardless of the center frequency f_0 and bandwidth B.

The idea is to do some processing at the channel input (actually implemented at the transmitter back-end) and at the channel output (implemented at the receiver front-end) in such a way that the new channel becomes a lowpass channel. The big picture is again that of Fig. 1.2.