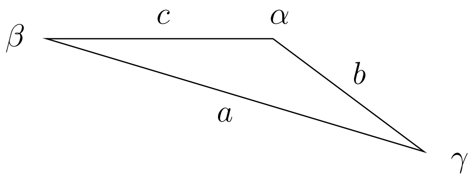
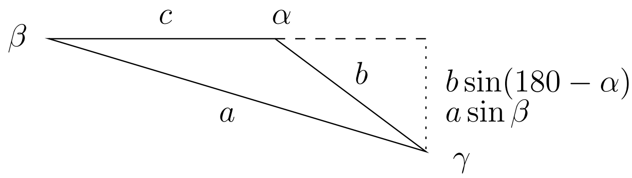
Appendix 2.D A Fact About Triangles

To determine an exact expression of the probability of error, in Example 7 we use the following fact about triangles.





For a triangle with edges a, b, c and angles α, β, γ (see the figure), the following relationship holds:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}. \tag{2.57}$$

To prove the equality relating a and b we project the common vertex γ onto the extension of the segment connecting the other two edges (α and β). This projection gives rise to two triangles that share a common edge whose length can be written as $a \sin \beta$ and as $b \sin(180 - \alpha)$ (see right figure). Using $b \sin(180 - \alpha) = b \sin \alpha$ leads to $a \sin \beta = b \sin \alpha$. The second equality is proved similarly. □

Appendix 2.E Inner Product Spaces

Vector Space

We assume that you are familiar with vector spaces. In this Chapter 2 we will be dealing with the vector space of n-tuples over $\mathbb{R}$ but later we will need both the vector space of n-tuples over $\mathbb{C}$ and the vector space of finite-energy complex-valued functions. So to be as general as needed we assume that the vector space is over the field of complex numbers, in which case it is called a *complex vector space*. When the scalar field is $\mathbb{R}$, the vector space is called a *real vector space*.

Inner Product Space

Given a vector space and nothing more, one can introduce the notion of a basis for the vector space, but one does not have the tool needed to define an orthonormal basis. Indeed the axioms of a vector space say nothing about geometric ideas such as "length" or "angle." To remedy, one endows the vector space with the notion of inner product.

Definition 29. *Let V be a vector space over $\mathbb{C}$. An inner product on V is a function that assigns to each ordered pair of vectors α, β in V a scalar $\langle \alpha, \beta \rangle$ in $\mathbb{C}$ in such a way* *that for all* α, β, γ *in* $\mathcal{V}$ *and all scalars* c *in* $\mathbb{C}$

(a) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$
$\langle c\alpha, \beta \rangle = c\langle \alpha, \beta \rangle$;

(b) $\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle^*$; *(Hermitian Symmertry)*

(c) $\langle \alpha, \alpha \rangle \geq 0$ *with equality iff* $\alpha = 0$.

It is implicit in (c) that $\langle \alpha, \alpha \rangle$ *is real for all* $\alpha \in \mathcal{V}$. *From (a) and (b), we obtain an additional property*

(d) $\langle \alpha, \beta + \gamma \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle$
$\langle \alpha, c\beta \rangle = c^* \langle \alpha, \beta \rangle$.

Notice that the above definition is also valid for a vector space over the field of real numbers but in this case the complex conjugates appearing in (b) and (d) are superfluous; however, over the field of complex numbers they are necessary for the consistency of the conditions. Without these complex conjugates, for any $\alpha \neq 0$ we would have the contradiction:

$$0 < \langle i\alpha, i\alpha \rangle = -1\langle \alpha, \alpha \rangle < 0,$$

where the first inequality follows from condition (c) and the fact that $i\alpha$ is a valid vector, and the equality follows from (a) and (d) (without the complex conjugate).

On $\mathbb{C}^n$ there is an inner product that is sometimes called the *standard inner product*. It is defined on $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ by

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_j a_j b_j^*.$$

On $\mathbb{R}^n$, the standard inner product is often called the dot or scalar product and denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$. Unless explicitly stated otherwise, over $\mathbb{R}^n$ and over $\mathbb{C}^n$ we will always assume the standard inner product.

An *inner product space* is a real or complex vector space, together with a specified inner product on that space. We will use the letter $\mathcal{V}$ to denote a generic inner product space.

Example 30. *The vector space* $\mathbb{R}^n$ *equipped with the dot product is an inner product space and so is the vector space* $\mathbb{C}^n$ *equipped with the standard inner product.* □

By means of the inner product we introduce the notion of length, called *norm*, of a vector α, via

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}.$$

Using linearity, we immediately obtain that the *squared norm* satisfies

$$\|\alpha \pm \beta\|^2 = \langle \alpha \pm \beta, \alpha \pm \beta \rangle = \|\alpha\|^2 + \|\beta\|^2 \pm 2Re\{\langle \alpha, \beta \rangle\}. \tag{2.58}$$

The above generalizes $(a \pm b)^2 = a^2 + b^2 \pm 2ab, a, b \in \mathbb{R}$, and $| a \pm b |^2 = | a |^2 + | b |^2 \pm 2Re\{ab\}, a, b \in \mathbb{C}$.

EXAMPLE 31. *Consider the vector space V spanned by a finite collection of complex-valued finite-energy signals, where addition of vectors and multiplication of a vector with a scalar (in $\mathbb{C}$) are defined in the obvious way. You should verify that the axioms of a vector space are fulfilled. This includes showing that the sum of two finite-energy signals is a finite-energy signal. The standard inner product for this vectors space is defined as*

$$\langle \alpha, \beta \rangle = \int \alpha(t)\beta^*(t)dt$$

which implies the norm

$$\|\alpha\| = \sqrt{\int |\alpha(t)|^2 dt}.$$

□

EXAMPLE 32. *The previous example extends to the inner product space $\mathcal{L}_2$ of all complex-valued finite-energy functions. This is an infinite dimensional inner product space and to be careful one has to deal with some technicalities that we will just mention here. (You may skip the rest of this example if you wish without loosing anything important for the sequel). If α and β are two finite-energy functions that are identical except on a countable number of points, then $\langle \alpha - \beta, \alpha - \beta \rangle = 0$ (the integral is over a function that vanishes except for a countable number of points). The definition of inner product requires that $\alpha - \beta$ be the zero vector. This seems to be in contradiction with the fact that $\alpha - \beta$ is non-zero on a countable number of point. To deal with this apparent contradiction one can define vectors to be equivalence classes of finite-energy functions. In other words, if the norm of $\alpha - \beta$ vanishes then α and β are considered to be the same vector and $\alpha - \beta$ is seen as a zero vector. This equivalence may seem artificial at first but it is actually consistent with the reality that if $\alpha - \beta$ has zero energy then no instrument will be able to distinguish between α and β. The signal captured by the antenna of a receiver is finite energy, thus in $\mathcal{L}_2$. It is for this reason that we are interested in $\mathcal{L}_2$. However, as we will see, the receiver may obtain a sufficient statistics by projecting the received signal on a finite-dimention subspace of $\mathcal{L}_2$. During our brief exposures with $\mathcal{L}_2$ we will not be confronted with the subtle issues we have just mentioned.* □

THEOREM 33. *If $\mathcal{V}$ is an inner product space, then for any vectors α, β in $\mathcal{V}$ and any scalar c,*

(a) $\|c\alpha\| = |c|\|\alpha\|$

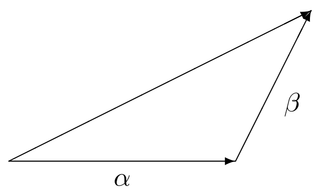
(b) $\|\alpha\| \geq 0$ *with equality iff* $\alpha = 0$

(c) $|\langle \alpha, \beta \rangle| \leq \|\alpha\|\|\beta\|$ *with equality iff* $\alpha = c\beta$ *for some* c.
(Cauchy-Schwarz inequality)

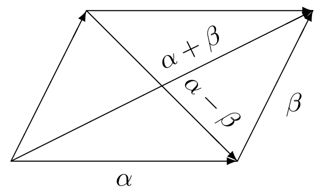
(d) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ *with equality iff* $\alpha = c\beta$ *for some non-negative* $c \in \mathbb{R}$.
(Triangle inequality)

(e) $\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$
(Parallelogram equality)

Proof. *Statements (a) and (b) follow immediately from the definitions. We postpone the proof of the Cauchy-Schwarz inequality to Example 35 since it will be more insightful once we have defined the concept of a projection. To prove the triangle inequality we use (2.58) and the Cauchy-Schwarz inequality applied to* $Re\{\langle\alpha,\beta\rangle\} \leq |\langle\alpha,\beta\rangle|$ *to prove that* $\|\alpha+\beta\|^2 \leq (\|\alpha\| + \|\beta\|)^2$. *You should verify that* $Re\{\langle\alpha,\beta\rangle\} \leq |\langle\alpha,\beta\rangle|$ *holds with equality iff* $\alpha = c\beta$ *for some non-negative* $c \in \mathbb{R}$. *Hence this condition is necessary for the triangle inequality to hold with equality. It is also sufficient since then also the Cauchy-Schwarz inequality holds with equality. The parallelogram equality follows immediately from (2.58) used twice, once with each sign.* □



Triangle inequality



Parallelogram equality

At this point we could use the inner product and the norm to define the angle between two vectors but we don't have any use for that. Instead, we will make frequent use of the notion of orthogonality. Two vectors α and β are defined to be *orthogonal* if $\langle\alpha,\beta\rangle = 0$.

Theorem 34. (Pythagorean Theorem) *If* α *and* β *are orthogonal vectors in* $\mathcal{V}$, *then*

$$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2.$$

Proof. The Pythagorean theorem follows immediately from the equality $\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2Re\{\langle\alpha,\beta\rangle\}$ and the fact that $\langle\alpha,\beta\rangle = 0$ by definition of orthogonality. □

Given two vectors $\alpha, \beta \in \mathcal{V}$, $\beta \neq 0$, we define the *projection of* α *on* β as the vector $\alpha_{|\beta}$ collinear to β (i.e. of the form $c\beta$ for some scalar c) such that $\alpha_{\perp\beta} = \alpha - \alpha_{|\beta}$ is orthogonal to β. Using the definition of orthogonality, what we want is

$$0 = \langle\alpha_{\perp\beta}, \beta\rangle = \langle\alpha - c\beta, \beta\rangle = \langle\alpha,\beta\rangle - c\|\beta\|^2.$$

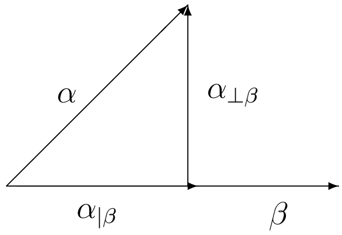
Solving for c we obtain $c = \frac{\langle\alpha,\beta\rangle}{\|\beta\|^2}$. Hence

$$\alpha_{|\beta} = \frac{\langle\alpha,\beta\rangle}{\|\beta\|^2}\beta \qquad \text{and} \qquad \alpha_{\perp\beta} = \alpha - \alpha_{|\beta}.$$

The projection of α on β does not depend on the norm of β. To see this let $\beta = b\psi$ for some $b \in \mathbb{C}$. Then

$$\alpha_{|\beta} = \langle\alpha,\psi\rangle\psi = \alpha_{|\psi},$$

regardless of b. The norm of the projection is $\langle\alpha,\psi\rangle = \langle\alpha,\beta\rangle/\|\beta\|$.



Projection of α on β

Any non-zero vector β defines a *hyperplane* by the relationship

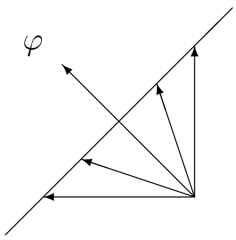
$$\{\alpha \in \mathcal{V} : \langle \alpha, \beta \rangle = 0\}\,.$$

It is the set of vectors that are orthogonal to β. A hyperplane always contains the zero vector.

An *affine space*, defined by a vector β and a scalar c, is an object of the form

$$\{\alpha \in \mathcal{V} : \langle \alpha, \beta \rangle = c\}\,.$$

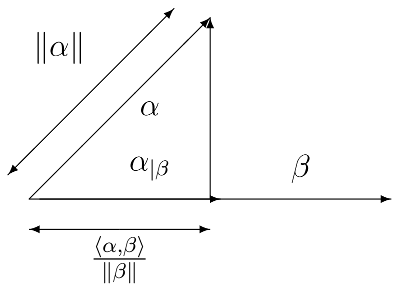
The defining vector and scalar are not unique, unless we agree that we use only normalized vectors to define hyperplanes. By letting $\varphi = \frac{\beta}{\|\beta\|}$, the above definition of affine plane may equivalently be written as $\{\alpha \in \mathcal{V} : \langle \alpha, \varphi \rangle = \frac{c}{\|\beta\|}\}$ or even as $\{\alpha \in \mathcal{V} : \langle \alpha - \frac{c}{\|\beta\|}\varphi, \varphi \rangle = 0\}$. The first shows that at an affine plane is the set of vectors that have the same projection $\frac{c}{\|\beta\|}\varphi$ on φ. The second form shows that the affine plane is a hyperplane translated by the vector $\frac{c}{\|\beta\|}\varphi$. Some authors make no distinction between affine planes and hyperplanes. In that case both are called hyperplane.



Affine plane defined by φ.

Now it is time to prove the Cauchy-Schwarz inequality stated in Theorem 33. We do it as an application of a projection.

Example 35. (Proof of the Cauchy-Schwarz Inequality). *The Cauchy-Schwarz inequality says states that for any $\alpha, \beta \in \mathcal{V}$, $|\langle \alpha, \beta \rangle| \le \|\alpha\|\|\beta\|$ with equality iff $\alpha = c\beta$ for some scalar $c \in \mathbb{C}$. The statement is obviously true if $\beta = 0$. Assume $\beta \neq 0$ and write $\alpha = \alpha_{|\beta} + \alpha_{\perp\beta}$. The Pythagorean theorem states that $\|\alpha\|^2 = \|\alpha_{|\beta}\|^2 + \|\alpha_{\perp\beta}\|^2$. If we drop the second term, which is always nonnegative, we obtain $\|\alpha\|^2 \ge \|\alpha_{|\beta}\|^2$ with equality iff α and β are collinear. From the definition of projection, $\|\alpha_{|\beta}\|^2 = \frac{|\langle \alpha, \beta \rangle|^2}{\|\beta\|^2}$. Hence $\|\alpha\|^2 \ge \frac{|\langle \alpha, \beta \rangle|^2}{\|\beta\|^2}$ with equality equality iff α and β are collinear. This is the Cauchy-Schwarz inequality.* $\square$



The Cauchy-Schwarz inequality

Every finite-dimensional vector space has a basis. If $\beta_1, \beta_2, \ldots, \beta_n$ is a basis for the inner product space $\mathcal{V}$ and $\alpha \in \mathcal{V}$ is an arbitrary vector, then there are scalars $a_1, \ldots, a_n$ such that $\alpha = \sum a_i \beta_i$ but finding them may be difficult. However, finding the coefficients of a vector is particularly easy when the basis is orthonormal.

A basis $\varphi_1, \varphi_2, \ldots, \varphi_n$ for an inner product space $\mathcal{V}$ is orthonormal if

$$\langle \varphi_i, \varphi_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Finding the i-th coefficient a_i of an orthonormal expansion $\alpha = \sum a_i \psi_i$ is immediate. It suffices to observe that all but the ith term of $\sum a_i \psi_i$ are orthogonal to ψ_i and that the inner product of the ith term with ψ_i yields a_i. Hence if $\alpha = \sum a_i \psi_i$ then

$$a_i = \langle \alpha, \psi_i \rangle.$$

Observe that a_i is the norm of the projection of α on ψ_i. This should not be surprising given that the ith term of the orthonormal expansion of α is collinear to ψ_i and the sum of all the other terms are orthogonal to ψ_i.

There is another major advantage of working with an orthonormal basis. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are the n-tuples of coefficients of the expansion of α and β with respect to the same orthonormal basis then

$$\langle \alpha, \beta \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \rangle$$

where the right hand side inner product is with respect to the standard inner product. Indeed

$$\begin{aligned}
\langle \alpha, \beta \rangle &= \langle \sum a_i \psi_i, \sum_j b_j \psi_j \rangle \\
&= \sum a_i \langle \psi_i, \sum_j b_j \psi_j \rangle \\
&= \sum a_i \langle \psi_i, b_i \varphi_i \rangle \\
&= \sum a_i b_i^* \\
&= \langle \boldsymbol{a}, \boldsymbol{b} \rangle.
\end{aligned}$$

Letting $\beta = \alpha$ the above implies also

$$\|\alpha\| = \|\boldsymbol{a}\|,$$

where the right hand side is the standard norm $\|a\| = \sum |a_i|^2$.

An orthonormal set of vectors $\psi_1, \ldots, \psi_n$ of an inner product space $\mathcal{V}$ is a linearly independent set. Indeed $0 = \sum a_i \psi_i$ implies $a_i = \langle 0, \psi_i \rangle = 0$. By normalizing the vectors and recomputing the coefficients one can easily extend this reasoning to a set of orthogonal (but not necessarily orthonormal) vectors $\alpha_1, \ldots, \alpha_n$. They too must be linearly independent.

The idea of a projection on a vector generalizes to a projection on a subspace. If $\mathcal{W}$ is a subspace of an inner product space $\mathcal{V}$, and $\alpha \in \mathcal{V}$, the projection of α on $\mathcal{W}$ is defined to be a vector $\alpha_{|\mathcal{W}} \in \mathcal{W}$ such that $\alpha - \alpha_{|\mathcal{W}}$ is orthogonal to all vectors in $\mathcal{W}$. If $\psi_1, \ldots, \psi_m$ is an orthonormal basis for $\mathcal{W}$ then the condition that $\alpha - \alpha_{|\mathcal{W}}$ is orthogonal to all vectors of $\mathcal{W}$ implies $0 = \langle \alpha - \alpha_{|\mathcal{W}}, \psi_i \rangle = \langle \alpha, \psi_i \rangle - \langle \alpha_{|\mathcal{W}}, \psi_i \rangle$. This shows that $\langle \alpha, \psi_i \rangle = \langle \alpha_{|\mathcal{W}}, \psi_i \rangle$. The right side of this equality is the i-th coefficient of the orthonormal expansion of $\alpha_{|\mathcal{W}}$ with respect to the orthonormal basis. This proves that

$$\alpha_{|\mathcal{W}} = \sum_{i=1}^{m} \langle \alpha, \psi_i \rangle \psi_i$$

is the unique projection of α on $\mathcal{W}$.

THEOREM 36. *Let $\mathcal{V}$ be an inner product space and let $\beta_1, \ldots, \beta_n$ be any collection of linearly independent vectors in $\mathcal{V}$. Then one may construct orthogonal vectors $\alpha_1, \ldots, \alpha_n$ in $\mathcal{V}$ such that they form a basis for the subspace spanned by $\beta_1, \ldots, \beta_n$.*

Proof. The proof is constructive via a procedure known as the Gram-Schmidt orthogonalization procedure. First let $\alpha_1 = \beta_1$. The other vectors are constructed inductively as follows. Suppose $\alpha_1, \ldots, \alpha_m$ have been chosen so that they form an orthogonal basis for the subspace $\mathcal{W}_m$ spanned by $\beta_1, \ldots, \beta_m$. We choose the next vector as

$$\alpha_{m+1} = \beta_{m+1} - \beta_{m+1|\mathcal{W}_m}, \tag{2.59}$$

where $\beta_{m+1|\mathcal{W}_m}$ is the projection of β_{m+1} on $\mathcal{W}_m$. By definition, α_{m+1} is orthogonal to every vector in $\mathcal{W}_m$, including $\alpha_1, \ldots, \alpha_m$. Also, $\alpha_{m+1} \neq 0$ for otherwise β_{m+1} contradicts the hypothesis that it is lineary independent of $\beta_1, \ldots, \beta_m$. Therefore $\alpha_1, \ldots, \alpha_{m+1}$ is an orthogonal collection of nonzero vectors in the subspace $\mathcal{W}_{m+1}$ spanned by $\beta_1, \ldots, \beta_{m+1}$. Therefore it must be a basis for $\mathcal{W}_{m+1}$. Thus the vectors $\alpha_1, \ldots, \alpha_n$ may be constructed one after the other according to (2.59). □

COROLLARY 37. *Every finite-dimensional vector space has an orthonormal basis.*

Proof. Let $\beta_1, \ldots, \beta_n$ be a basis for the finite-dimensionall inner product space $\mathcal{V}$. Apply the Gram-Schmidt procedure to find an orthogonal basis $\alpha_1, \ldots, \alpha_n$. Then $\psi_1, \ldots, \psi_n$, where $\psi_i = \frac{\alpha_i}{\|\alpha_i\|}$, is an orthonormal basis. □

Gram-Schmidt Orthonormalization Procedure

We summarize the Gram-Schmidt procedure, modified so as to produce orthonormal vectors. If $\beta_1, \ldots, \beta_n$ is a linearly independent collection of vectors in the inner product space $\mathcal{V}$ then we may construct a collection $\psi_1, \ldots, \psi_n$ that forms an orthonormal basis for the subspace spanned by $\beta_1, \ldots, \beta_n$ as follows: we let $\psi_1 = \frac{\beta_1}{\|\beta\|}$ and for $i = 2, \ldots, n$ we choose

$$\alpha_i = \beta_i - \sum_{j=1}^{i-1} \langle \beta_i, \psi_j \rangle \psi_j$$
$$\psi_i = \frac{\alpha_i}{\|\alpha_i\|}.$$

The following table gives an example of the Gram-Schmidt procedure.

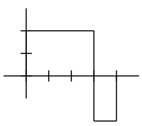
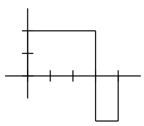
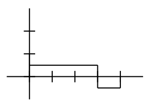
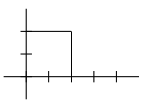
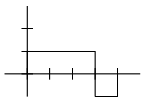
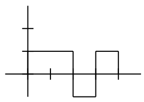
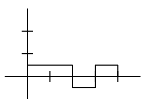
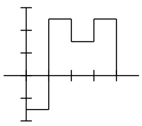
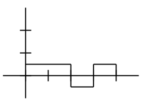
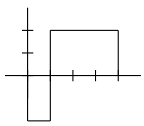
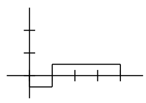
i	β_i	$\langle \beta_i, \psi_j \rangle$ $j < i$	$\beta_{i\|\mathcal{W}_{i-1}}$	$\alpha_i = \beta_i - \beta_{i\|\mathcal{W}_{i-1}}$	$\|\alpha_i\|$	ψ_i	$\boldsymbol{\beta}_i$
1		-	-		2		$\begin{pmatrix}2\\0\\0\end{pmatrix}$
2		1			1		$\begin{pmatrix}1\\1\\0\end{pmatrix}$
3		$0, 1$			4		$\begin{pmatrix}0\\1\\4\end{pmatrix}$

Table 2.1: Application of the Gram-Schmidt orthonormalization procedure. Axes are marked with unit length intervals.

Chapter 3

Communication Across the Waveform AWGN Channel

3.1 Introduction

In the previous chapter we have learned how to communicate across the Vector AWGN (Additive White Gaussian Noise) channel. Given a transmitter for that channel, we now know what a receiver that minimizes the error probability should do and how to evaluate or bound the resulting error probability. In this chapter we will deal with a channel model which is closer to reality, namely the Waveform AWGN channel. Apart form the channel model, the main objectives of this and the previous chapters are the same: understand what a receiver should do to minimize the error probability and learn techniques to evaluate the receiver performance. We will also learn that the transmitter and the receiver for the waveform channel may be obtained as natural extensions from the transmitter and receiver for the vector channel studied in the previous chapter. The extension of the transmitter is the Waveform Generator and that of the receiver is the Baseband Front-End, both shown in Figure 1.2. No new technique will be needed to evaluate the error probability.

The starting point for this chapter is the system model shown in Figure 3.1. As usual, we assume that the channel is given (which means that we know the power spectral density N_0 of the white Gaussian noise), and that we have to design the Transmitter (TX) and the Receiver (RX).

The operation of the Waveform Transmitter is similar to that of the Vector Transmitter of the previous chapter except that the output $S(t)$ is now an element of a set of m finite-energy waveforms

$$S(t) \in \{s_0(t), \ldots, s_{m-1}(t)\} \subset \mathcal{L}_2.$$

The task of the receiver is to implement a ML decision rule for the following hypothesis

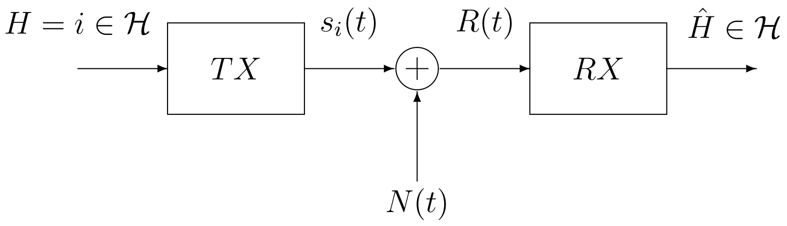


Figure 3.1: Communication across the AWGN channel.

testing problem:

$$H = i: \quad R(t) = s_i(t) + N(t),$$

where $N(t)$ is a zero-mean white Gaussian noise process with (two-sided) power spectral density $N_0/2$. (N_0 is the one-sided power spectral density, namely what one would measure with an instrument.) The source picks the index i according to some probability $P_H(i)$ (typically uniform) over the message set $\mathcal{H}$.

We will see that, without loss of generality, we may (and should) think of the transmitter as consisting of a part that maps the message H into an n-tuple, as in the previous chapter, followed by a *waveform generator* that maps the n-tuple into a waveform. Similarly, we will see that the receiver may consist of a *front-end* that takes the channel output and produces an n-tuple that is a sufficient statistic. From the waveform generator input to the receiver front-end output, we see a *vector channel* of the kind studied in the previous chapter. Hence, we know already what an optimal receiver should do with the sufficient statistic produced by the receiver front end.

In this chapter we assume familiarity with the linear space $\mathcal{L}_2$ of finite energy functions and with the concept of white Gaussian noise (WGN) process. Throughout the chapter we will assume that the set $\{s_i(t) : i \in \mathcal{H}\}$ is given. The problem of choosing this set conveniently will be studied in subsequent chapters.

3.2 The Binary Equiprobable Case

We start with the binary hypothesis case since it allows us to focus on the essential. Generalizing to M hypothesis will be straightforward.

There are two hypotheses. When $H = i$, $i \in \mathcal{H} = \{0, 1\}$, we send the waveform $s_i(t)$. To avoid distractions, we assume $P_H(1) = 1/2$. The receiver observes $R(t) = s_i(t) + N(t)$, where $N(t)$ is white Gaussian noise of constant power spectral density $N_0/2$. We are interested in the receiver that minimizes the probability of error.

3.2.1 Sufficient Statistics via Projections

The strategy is to reduce the new (waveform) hypothesis testing problem to the familiar problem where we observe n-tuples.

The key idea is that waveforms of an inner product space can be represented by n-tuples. Which inner product space should we work with? We would like to use the smallest possible one, i.e., the one spanned by s_0 and s_1. Let us call this space $\mathcal{W}$. There is a potential problem though: the noise is not in $\mathcal{W}$, hence $R = s_i + N$ is not in $\mathcal{W}$ either.

If we project the received waveform onto $\mathcal{W}$, we obtain a waveform $Y = R_{|\mathcal{W}}$ which consists of R minus the portion of the noise which is orthogonal to $\mathcal{W}$ (see Figure 3.2). The intuition is that, in so doing, we remove just noise from the received waveform. More formally, a MAP decision rule based on $R_{|\mathcal{W}}$ results in the same probability of error as one based on R iff $R_{|\mathcal{W}}$ is a *sufficient statistic* or, equivalently, if the portion $N_\perp$ of the noise being removed is *irrelevant*. This is the case if, conditioned on $R_{|\mathcal{W}}$ and H, the pdf of $N_\perp$ does not depend on H. But this is indeed the case since any finite collection of samples from $N_\perp$ is independent of both H and $R_{|\mathcal{W}}$.

Based on the above argument, we consider $R_{|\mathcal{W}}$ as the observable. To be consistent with our notation we will use Y for $R_{|\mathcal{W}}$ and Z for $N_{|\mathcal{W}}$.

We can now restate our hypothesis testing problem:

$$H = i : \qquad Y = s_i + Z.$$

We hope that this is a progress since Y, s_i, and Z are all in $\mathcal{W}$. After choosing a basis $\{\psi_1, \psi_2\}$ for $\mathcal{W}$, e.g. via the Gram Schmidt procedure on s_0 and s_1, the corresponding n-tuples $\boldsymbol{Y}$, $\boldsymbol{s}_i$, and $\boldsymbol{Z}$ are well defined. Specifically:

$$\begin{aligned} \boldsymbol{Y} &= (Y_1, Y_2)^T \text{ where} \\ Y_i &= \langle R, \psi_i \rangle, \quad i = 1, 2, \end{aligned}$$

$$\begin{aligned} \boldsymbol{s}_k &= (s_{k1}, s_{k2})^T \text{ where} \\ s_{ki} &= \langle s_k, \psi_i \rangle, \quad i = 1, 2, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{Z} &= (Z_1, Z_2)^T \text{ where} \\ Z_i &= (N, \psi_i), \quad i = 1, 2. \end{aligned}$$

Hereafter we will use the following convention. We use lowercase fonts for deterministic vectors in $\mathcal{L}_2$ such as s_i. For random vectors in $\mathcal{L}_2$ we use capital letters, such as S. The corresponding $n$tuples will be denoted with bold letters such as $\boldsymbol{s}_i$ and $\boldsymbol{S}$.

The hypothesis testing problem based on $\boldsymbol{Y}$ is now the familiar

$$H = i : \qquad \boldsymbol{Y} = \boldsymbol{s}_i + \boldsymbol{Z} \qquad \boldsymbol{Z} \sim \mathcal{N}(0, \frac{N_0}{2} I_2).$$

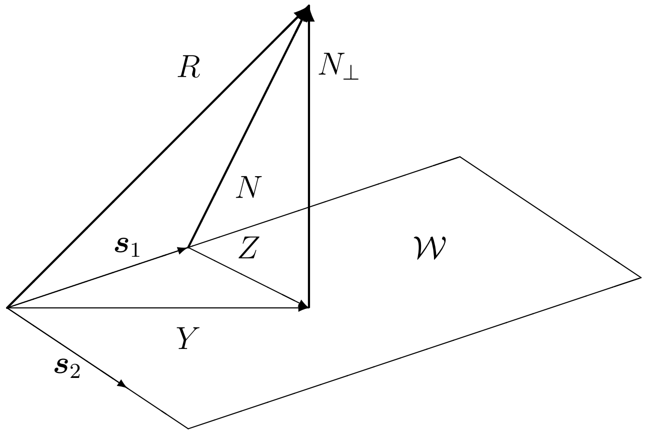


Figure 3.2: Projection of R onto $\mathcal{W}$. It is assumed that $H = i$. Thinner vectors are in $\mathcal{W}$

3.2.2 Optimal Test

The test that minimizes the error probability is the ML decision rule:

$$\|\boldsymbol{y} - \boldsymbol{s}_0\|^2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \|\boldsymbol{y} - \boldsymbol{s}_1\|^2.$$

As usual, ties may be resolved either way.

3.2.3 Receiver Structures

In this section we deal with receiver structures. There are various ways to implement the receiver since :

(a) the ML test can be rewritten in various ways.

(b) there are two basic ways to implement a projection;

Hereafter is a list of equivalent ML tests. Each of them either suggests a viable implementation or provides valuable insight . The tests are:

$$\|\boldsymbol{y}-\boldsymbol{s}_0\| \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \|\boldsymbol{y}-\boldsymbol{s}_1\| \qquad \text{(T1)}$$

$$\langle \boldsymbol{y}, \boldsymbol{s}_1\rangle - \frac{\|\boldsymbol{s}_1\|^2}{2} \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \langle \boldsymbol{y}, \boldsymbol{s}_0\rangle - \frac{\|\boldsymbol{s}_0\|^2}{2} \qquad \text{(T2)}$$

$$\langle R, s_1\rangle - \frac{\|s_1\|^2}{2} \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \langle R, s_0\rangle - \frac{\|s_0\|^2}{2} \qquad \text{(T3)}$$

Test (T1) is the test derived in the previous section after taking the square root on both sides. Since the square root of a nonnegative number is a monotonic operation, the test outcome remains unchanged. Test (T1) is useful to visualize decoding regions and to compute the probability of error. It says that the decoding region of $\boldsymbol{s}_0$ is the set of $\boldsymbol{y}$ that are closer to $\boldsymbol{s}_0$ than to $\boldsymbol{s}_1$.

Figure 3.3 shows the block diagram of a receiver inspired by (T1). The receiver front-end maps R into $\boldsymbol{Y} = (Y_1, Y_2)$. This part of the receiver deals with waveforms and in the past it has been implemented via analog circuitry. The slicer implements the test (T1). We will refer often to the slicer. It is a conceptual device that knows the decoding regions and checks which decoding region contains $\boldsymbol{y}$. The slicer shown in the Figure 3.3 assumes *antipodal signals*, i.e., $s_0 = -s_1$, and $\psi_1 = s_1/\|s_1\|$. In this case the signal space is one-dimensional and Y_2 is irrelevant.

A slicer for a 2-dimensional signal space spanned by orthogonal signals s_0 and s_1 is shown in Figure 3.4, where we defined $\psi_1 = s_0/\|s_0\|$ and $\psi_2 = s_1/\|s_1\|$.

Once we have a description of the decoding regions, it is conceptually easy to compute the probability of error under each hypothesis. For instance, the probability of error given that $H = 0$ is the probability that an iid Gaussian random vector $\boldsymbol{Z}$ of variance $\frac{N_0}{2}$ in each component centered at $\boldsymbol{s}_0$ ends up in the decoding region of $\boldsymbol{s}_1$. This is the integral over the decoding region of $\boldsymbol{s}_1$ of a Gaussian p.d.f. centered at $\boldsymbol{s}_0$.

Depending on the shape of the decoding region of $\boldsymbol{s}_1$, carrying out the integration may or may not be easy. It is easy in general when the decoding region is bounded by perpendicular half-planes as in Figure 3.3 and 3.4. In this case the result can simply be expressed in terms of Q-functions. Explicit examples will be given later.

Perhaps the biggest advantage of test (T1) is the geometrical insight it gives as shown by the slicer. It is, however, not the most economical test in terms of number of steps

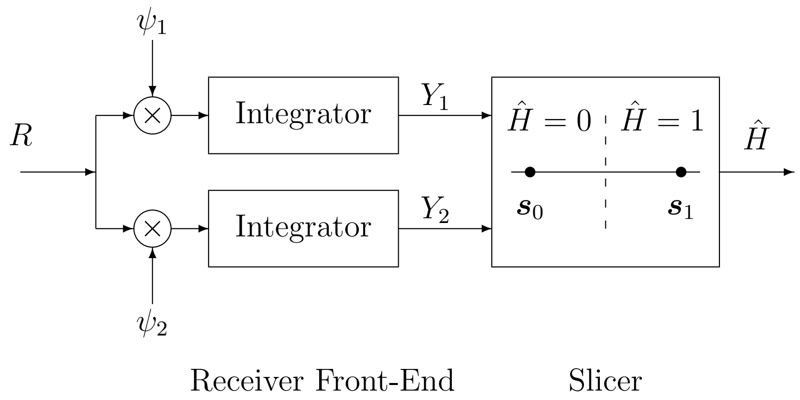


Figure 3.3: Implementation of test (T1). The front-end is based on correlators.

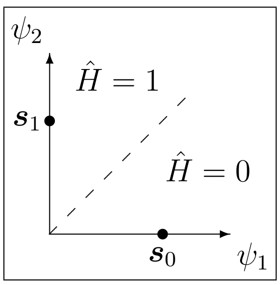


Figure 3.4: Slicer for two orthogonal signals

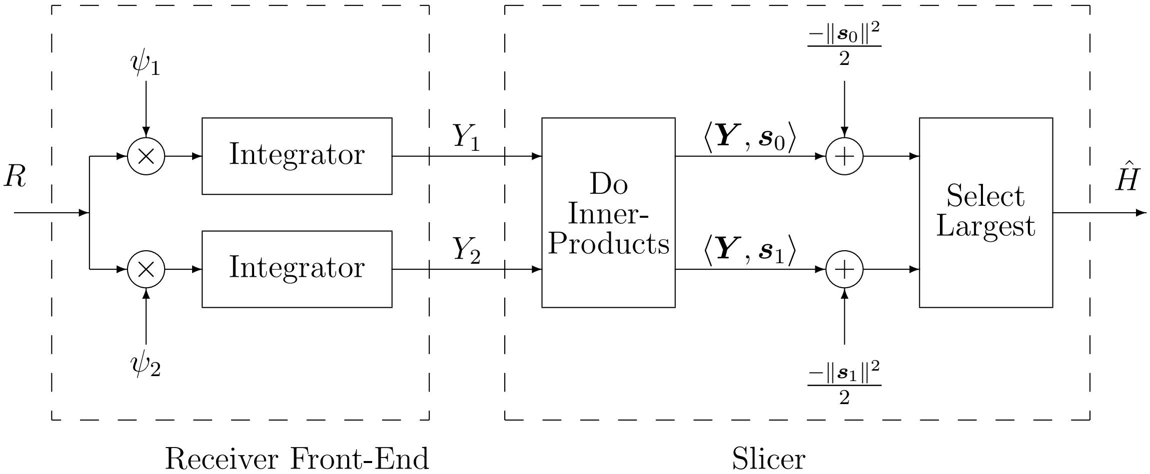


Figure 3.5: Receiver implementation following (T2)

needed to implement it verbatim.

Test (T2) is obtained from (T1) using the relationship

$$\begin{aligned}\|\boldsymbol{y} - \boldsymbol{s}_i\|^2 &= \langle \boldsymbol{y} - \boldsymbol{s}_i, \boldsymbol{y} - \boldsymbol{s}_i \rangle \\ &= \|\boldsymbol{y}\|^2 - 2Re\{\langle \boldsymbol{y}, \boldsymbol{s}_i \rangle\} + \|\boldsymbol{s}_i\|^2,\end{aligned}$$

after canceling out common terms, multiplying each side by $-1/2$, and using the fact that $a > b$ iff $-a < -b$. Test (T2) is implemented by the block diagram of Figure 3.5.

The added value of the slicer in Figure 3.5 is that its operation is completely specified in terms of easy-to-implement operations. However, the slicer in Figure 3.3 gives more geometrical insight.

Test (T3) is obtained from (T2) via Parseval's relationship and a bit more to account for the fact that projecting R onto s_i is the same as projecting Y. Specifically, for $i = 1, 2$,

$$\begin{aligned}\langle \boldsymbol{y}, \boldsymbol{s}_i \rangle &= \langle Y, s_i \rangle \\ &= \langle Y + N_\perp, s_i \rangle \\ &= \langle R, s_i \rangle.\end{aligned}$$

Test (T3) is implemented by the block diagram in Figure 3.6.

Even tough test (T2) and (T3) look similar, they differ fundamentally and practically. First of all (T3) does not require finding a basis for the signal space spanned by $s_i, i = 1, 2$. As a side benefit this proves that the receiver performance does not depend on the basis used to perform (T2) (or (T1) for that matter).

Second, Test (T2) requires an extra layer of computation, namely that needed to perform the inner products $\langle \boldsymbol{y}, \boldsymbol{s}_i \rangle$. This step comes for free in (T3).

However, the number of integrators needed in Figure 3.6 equals the number m of hypotheses (2 in our case), whereas that in Figure 3.5 equals to dimensionality n of the

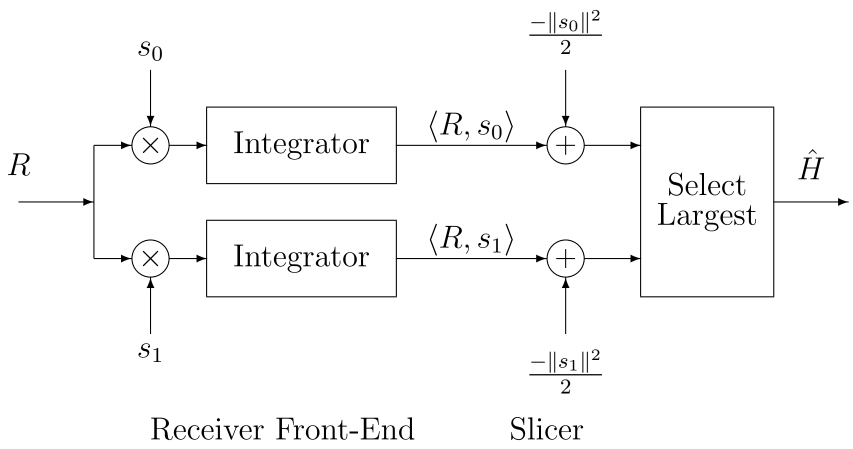


Figure 3.6: Receiver implementation following (T3)

signal space W. We know that $n \leq m$ and one can easily construct examples where equality holds or where $n \ll m$. In the latter case it is preferable to implement test (T2).

Each of the tests (T1), (T2), and (T3) can be implemented in two ways. One way is shown in Figs. 3.3, 3.5 and 3.6, respectively. The other way makes use of the fact that a projection

$$\langle R, s \rangle = \int R(t)s^*(t)dt$$

can always be implemented by means of a filter of impulse response $h(t) = s^*(T-t)$ as shown in Figure 3.7 (b), where T is an arbitrary delay selected in such a way as to make h a causal impulse response.

To verify that the implementation of Figure (3.7)(b) also leads to $\langle R, s \rangle$, we proceed as follows. Let y be the filter output when the input is R. If $h(t) = s^*(T-t)$, $t \in \mathbb{R}$, is the filter impulse response, then

$$y(t) = \int R(\alpha)\, h(t-\alpha)\, d\alpha = \int R(\alpha)\, s^*(T+\alpha-t)\, d\alpha.$$

At $t = T$ the output is

$$y(T) = \int R(\alpha)\, s^*(\alpha)\, d\alpha,$$

which is indeed $\langle R, s \rangle$ (by definition). The implementation of Figure 3.7(b) is referred to as *matched-filter implementation* of the receiver front-end.

In each of the receiver front ends shown in Figs. 3.3, 3.5 and 3.6, we can substitute matched filters for correlators.

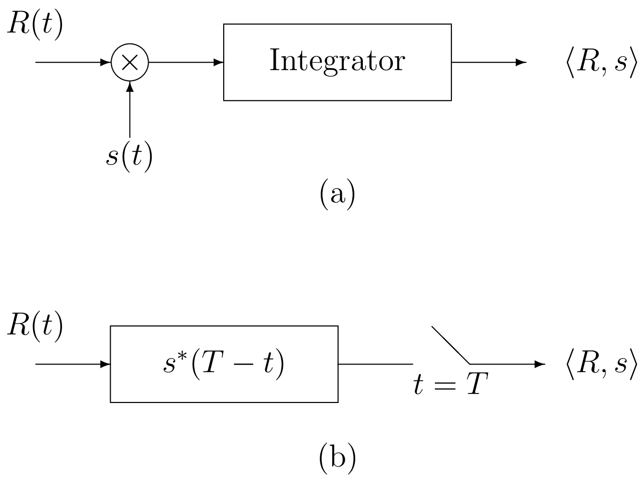


Figure 3.7: Two ways to implement the projection $\langle R, s\rangle$, namely via a "correlator" (a) and via a "matched filter" (b).

3.2.4 Probability of Error

Computing the probability of error is straightforward when we have only two hypotheses. From test (T1) we see that when $H = 0$ we make an error if $\boldsymbol{Y}$ is closer to $\boldsymbol{s}_1$ than to $\boldsymbol{s}_0$. This happens if the projection of the noise N in direction $\boldsymbol{s}_1 - \boldsymbol{s}_0$ exceeds $\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_0\|}{2}$. This event has probability.

$$P_e(0) = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_0\|}{2\sigma}\right)$$

where $\sigma^2 = \frac{N_0}{2}$ is the variance of the projection of the noise in any direction.

By symmetry, $P_e(1) = P_e(0)$. Hence

$$P_e = \frac{1}{2}P_e(1) + \frac{1}{2}P_e(0) = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_0\|}{\sqrt{2N_0}}\right),$$

where

$$\|\boldsymbol{s}_1 - \boldsymbol{s}_0\| = \|s_1 - s_0\| = \sqrt{\int [s_1(t) - s_0(t)]^2 dt}.$$

It is interesting to observe that the probability of error depends only on the distance $\|s_1 - s_0\|$ and not on the particular shape of the waveforms s_0 and s_1.

In the following example we represent a rectangular pulse by an indicator function.

Example 38. *Consider the following signal choices and verify that, in all cases, the signal space representation is* $\boldsymbol{s}_0 = (\sqrt{\mathcal{E}}, 0)^T$ *and* $\boldsymbol{s}_1 = (0, \sqrt{\mathcal{E}})^T$. *To reach this conclusion, it is* *enough to verify that* $\langle \boldsymbol{s}_i, \boldsymbol{s}_j \rangle = \mathcal{E}\delta_{ij}$ *, where* δ_{ij} *equals* 1 *if* $i = j$ *and* 0 *otherwise. This means that, in each case,* $\boldsymbol{s}_0$ *and* $\boldsymbol{s}_2$ *are orthogonal.*

Choice 1 (Rectangular Pulse Position Modulation) :

$$\begin{aligned} s_0(t) &= \sqrt{\frac{\mathcal{E}}{T}}\, 1_{[0,T]}(t) \\ s_1(t) &= \sqrt{\frac{\mathcal{E}}{T}}\, 1_{[T,2T]}(t). \end{aligned}$$

Rectangular pulses can easily be generated, e.g. by a switch. They are used to communicate binary symbols within a circuit. A drawback of rectangular pulses is that they have infinite support in the frequency domain.

Choice 2 (Frequency Shift Keying):

$$\begin{aligned} s_0(t) &= \sqrt{\frac{2\mathcal{E}}{T}} \sin\left(\pi k \frac{t}{T}\right) 1_{[0,T]}(t) \\ s_1(t) &= \sqrt{\frac{2\mathcal{E}}{T}} \sin\left(\pi l \frac{t}{T}\right) 1_{[0,T]}(t), \end{aligned}$$

where k *and* l *are positive integers,* $k \neq l$ *. With a large value of* k *and* l *, these signals could be used for wireless communication. As they are they also have infinite support in the frequency domain. Using the trigonometric identity* $\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ *, it is straightforward to verify that the signals are orthogonal.*

Choice 3 (Sinc Pulse Position Modulation):

$$\begin{aligned} s_0(t) &= \sqrt{\frac{\mathcal{E}}{T}} \operatorname{sinc}\left(\frac{t}{T}\right) \\ s_1(t) &= \sqrt{\frac{\mathcal{E}}{T}} \operatorname{sinc}\left(\frac{t-T}{T}\right) \end{aligned}$$

The biggest advantage of sinc pulses is that they have finite support in the frequency domain. This means that they have infinite support in the time domain. In practice one uses a truncated version of the time domain signal.

Choice 4 (Spread Spectrum):

$$\begin{aligned} s_0(t) &= \sqrt{\frac{1}{T}} \sum_{j=1}^{n} s_{0j} 1_{[0,\frac{T}{n}]}\left(t - j\frac{T}{n}\right) \\ s_1(t) &= \sqrt{\frac{1}{T}} \sum_{j=1}^{n} s_{1j} 1_{[0,\frac{T}{n}]}\left(t - j\frac{T}{n}\right) \end{aligned}$$

where $\underline{s}_0 = (s_{01}, \ldots, s_{0n})^T$ *and* $\underline{s}_1 = (s_{11}, \ldots, s_{1n})^T$ *are orthogonal and have square norm* $\mathcal{E}$ *. This signaling method is called spread spectrum. It uses much bandwidth but it has an inherent robustness with respect to interfering (non-white) signals.*

As a function of time, the above signal choices vary form choice to choice quite significantly. Nevertheless, as you should be able to convince yourself quickly, on an AWGN channel they all lead to the same probability of error.

3.3 The m-ary Case And The Vector Channel

The solution to the binary hypothesis testing problem derived thus far easily generalizes to the m-ary hypothesis testing problem that we re-formulate for convenience:

$$H = j : \qquad R = s_j + N, \qquad j \in \mathcal{H},$$

where $\mathcal{H} = \{1, 2, \ldots, m\}$, $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$ is the signal constellation which is assumed to be known to the receiver, N is white Gaussian noise, and $P_H(i)$ is the probability that hypothesis H_j, $j \in \mathcal{H}$, is selected. Here R, s_j, and N are functions of time.

We summarize the steps leading to the optimal receivers, leaving out details since they are handled as in the binary case.

First we assume that we have selected a basis $\{\psi_1, \psi_2, \ldots, \psi_n\}$ for the vector space spanned by $s_1, s_2, \ldots, s_m$, denoted $\mathcal{W} = \mathcal{W}\{s_1, s_2, \ldots, s_m\}$. Like for the binary case, it will turn out that an optimal receiver can be implemented without going through the step of finding a basis.

At the receiver we obtain a sufficient statistic by projecting the received signal R onto each of the basis vector. The result is:

$$\begin{aligned} \boldsymbol{Y} &= (Y_1, Y_2, \ldots, Y_n)^T \text{ where} \\ Y_i &= \langle R, \psi_i \rangle, \quad i = 1, \ldots, n. \end{aligned}$$

The signal $Y = \sum Y_i \psi_i$ differs from the received signal R by the component of the noise which is orthogonal to the signal space $\mathcal{W}$. Let $\boldsymbol{s}_i = (s_{i1}, \ldots, s_{in})^T$ be the n-tuple whose components are the coefficient of s_i with respect to the selected basis. Then

$$s_i = \sum_{j=1}^{n} s_{ij} \psi_j.$$

It is instructive to visualize the transmitter and the receiver as shown in Figure 3.8. As indicated in this figure, we may think of the cascade of the waveform generator, waveform channel, and receiver front end as of a *vector*[1] *channel*. The vector channel is a suitable channel model to describe (in mathematical terms) the statistical behavior of the channel output given the input. It is also suitable to derive the receiver that minimizes the probability of error. It is also the channel model of choice in information theory to derive the channel capacity. The channel capacity is the maximal rate at which it is possible to

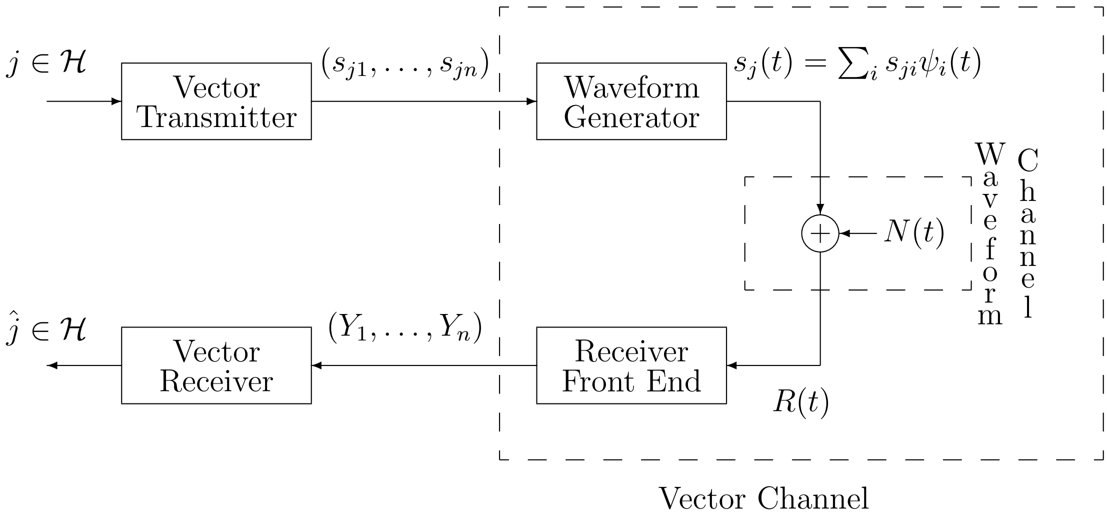


Figure 3.8: The waveform generator and receiver front-end transform the waveform channel into the vector channel.

transmit information in a reliable way across a physical channel by a suitable choice of the transmitter/receiver pair.

The Vector Receiver "sees" the vector hypothesis testing problem

$$H = j: \qquad \boldsymbol{Y} = \boldsymbol{s}_j + \boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{s}_j, \frac{N_0}{2} I_n)$$

studied in Chapter 2.

The receiver observes $\boldsymbol{y}$ and decides for $\hat{H} = j$ only if

$$P_H(j) f_{\boldsymbol{Y}}^{(j)}(\boldsymbol{y}) = \max_k \{P_H(k) f_{\boldsymbol{Y}}^{(k)}(\boldsymbol{y})\}$$

where we have introduce the notation $f_{\boldsymbol{Y}}^{(k)}(\boldsymbol{y})$ for $f_{\boldsymbol{Y}|H}(\boldsymbol{y}|k)$.

Any receiver that satisfies this decision rule minimizes the probability of error. If the maximum is not unique, the receiver may declare any of the hypotheses that achieves the maximum.

For the additive white Gaussian channel under consideration

$$f_{\boldsymbol{Y}}^{(j)}(\boldsymbol{y}) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{\|\boldsymbol{y} - \boldsymbol{s}_j\|^2}{2\sigma^2}\right)$$

where $\sigma^2 = \frac{N_0}{2}$.

[1]More appropriately, this channel should be called n*-tuple channel*.

Plugging into the above decoding rule, taking the log which is a monotonic function, multiplying by minus N_0, and canceling terms that do not depend on j, we obtain that a MAP decoder decides for one of the $j \in \mathcal{H}$ that minimizes

$$-N_0 \ln P_H(j) + \|\boldsymbol{y} - \boldsymbol{s}_j\|^2.$$

The expression should be compared to test (T1) of the previous section. The manipulations of $\|y - s_j\|^2$ that have led to test (T2) and (T3) are valid also here. In particular, the equivalent of (T2) consists of maximizing.

$$\langle \boldsymbol{y}, \boldsymbol{s}_j \rangle + c_j$$

where $c_j = \frac{1}{2}(N_0 \; \ln \; P_H(j) - \|s_j\|^2)$.

Finally, we can use Parseval's relationship to substitute $\langle R, s_j \rangle$ for $\langle \boldsymbol{Y}, \boldsymbol{s}_j \rangle$ and get rid of the need to find an orthonormal basis. This leads to the generalization of (T3), namely

$$\langle R, s_j \rangle + c_j.$$

Figure 3.9 shows three MAP receivers where the receiver front end is implemented via a bank of matched filters. Three alternative forms are obtained by using correlators instead of matched filters.

In the first figure, the slicer partitions $\mathbb{C}^n$ into decoding regions. The decoding region for $H = j$ is the set of points $\boldsymbol{y} \in \mathbb{C}^n$ for which

$$-N_0 \; \ln \; P_H(k) + \|\boldsymbol{y} - \boldsymbol{s}_k\|^2$$

is minimized when $k = j$.

Notice that in the first two implementations there are n matched filters, where n is the dimension of the signal space $\mathcal{W}$ spanned by $\mathcal{S}$, whereas in the third implementation the number of matched filters equals the number m of signals in $\mathcal{S}$. In general, $n \leq m$. Sometimes $n = m$. In this case the third implementation is preferable to the second since it does not require the weighing matrix and does not require finding a basis for $\mathcal{W}$. Sometimes $n = 1$ whereas m is large. Then the second implementation is preferable since it requires fewer filters.

3.4 Problems

Problem 1. (Matched Filter Implementation.)

In this problem, we consider the implementation of matched filter receivers. In particular, we consider Frequency Shift Keying (FSK) with the following signals:

$$s_j(t) \;=\; \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi \frac{n_j}{T} t, & \text{for } 0 \leq t \leq T, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

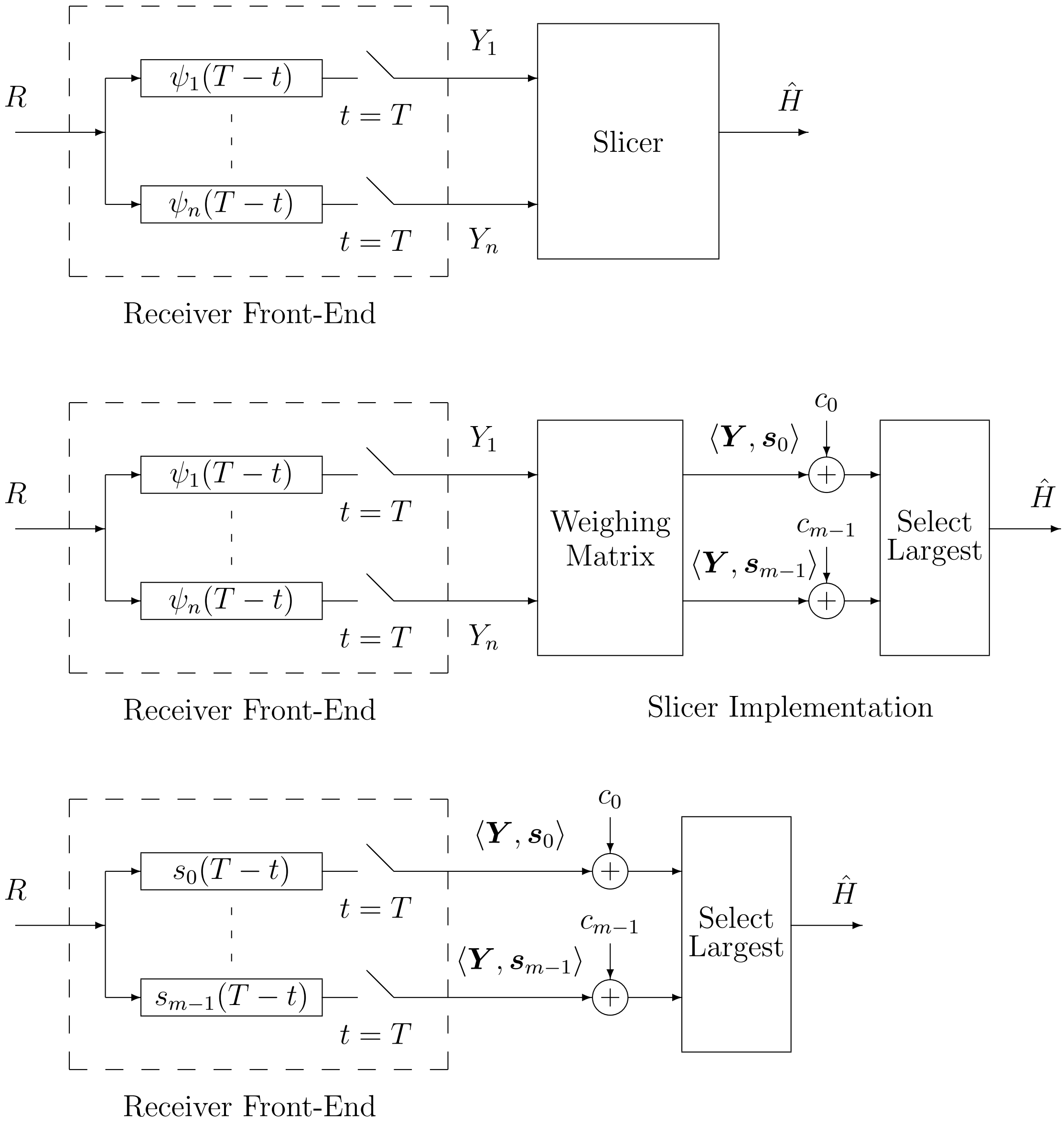


Figure 3.9: Three block diagrams of an optimal receiver. Each receiver front end may alternatively be implemented via correlators.

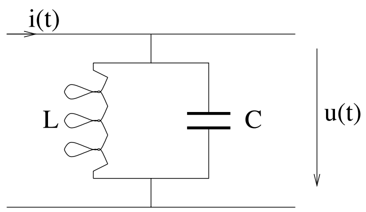
where $n_j \in \mathbb{Z}$ *and* $0 \le j \le m-1$. *Thus, the communications scheme consists of* m *signals* $s_j(t)$ *of different frequencies* $\frac{n_j}{T}$

(i) *Determine the impulse response* $h_j(t)$ *of the matched filter for the signal* $s_j(t)$. *Plot* $h_j(t)$.

(ii) *Sketch the matched filter receiver. How many matched filters are needed?*

(iii) *For* $-T \le t \le 3T$, *sketch the output of the matched filter with impulse response* $h_j(t)$ *when the input is* $s_j(t)$. *(*Hint: We recommend you to use Matlab.*)*

(iv) *Consider the following ideal resonance circuit:*



For this circuit, the voltage response to a unit impulse of current is

$$h(t) = \frac{1}{C}\cos\frac{t}{\sqrt{LC}}. \tag{3.2}$$

Show how this can be used to implement the matched filter for signal $s_j(t)$. *Determine how* L *and* C *should be chosen.* Hint: *Suppose that* $i(t) = s_j(t)$. *In that case, what is* $u(t)$ *?*

Problem 2. (On-Off Signaling)

Consider the following equiprobable binary hypothesis testing problem specified by:

$$\begin{aligned} H = 0 &: \; Y(t) = s(t) + N(t) \\ H = 1 &: \; Y(t) = N(t) \end{aligned}$$

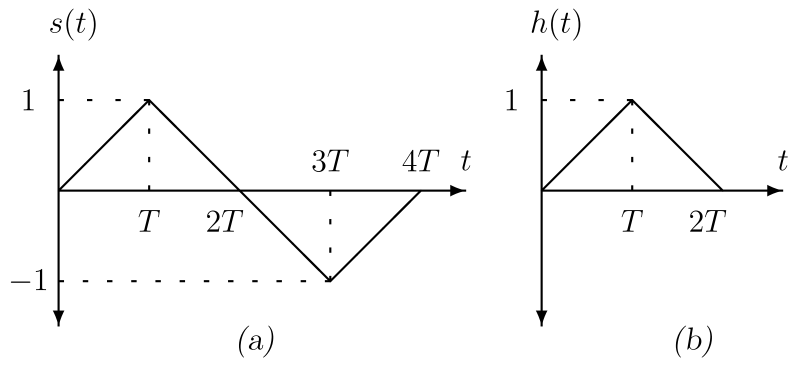
where $N(t)$ *is AWGN (Additive White Gaussian Noise) of power spectral density* $N_0/2$ *and* $s(t)$ *is the signal shown in the Figure (a) below.*

(a) *First consider a receiver that only observes* $Y(t_0)$ *for some fixed* t_0. *Does it make sense to choose* $\hat{H}$ *based on* $Y(t_0)$ *? Explain.*

(b) *Describe the maximum-likelihood receiver for the observable* $Y(t)$, $t \in \mathcal{R}$.

(c) *Determine the error probability for the receiver you described in* (b).

(d) *Can you realize your receiver of part* (b) *using a filter with impulse response* $h(t)$ *shown in Figure (b)?*



Problem 3. (Matched Filter Basics)

Consider a communication system that uses antipodal signals $S_i \in \{-1, 1\}$. *Using a fixed function* $h(t)$, *the transmitted waveform* $S(t)$ *is*

$$S(t) \quad = \quad \sum_{k=1}^{K} S_k \; h(t - kT).$$

The function $h(t)$ *and its shifts by multiples of* T *form an orthonormal set, i.e.,*

$$\int_{-\infty}^{\infty} h(t)h(t - kT)dt \quad = \quad \begin{cases} 0, & k \neq 0 \\ 1, & k = 0. \end{cases}$$

Hint: *You don't need Parts* (a) *and* (b) *to solve Part* (c).

(a) *Suppose* $S(t)$ *is filtered at the receiver by the matched filter with impulse response* $h(-t)$. *That is, the filtered waveform is* $R(t) = \int_{-\infty}^{\infty} S(\tau)h(\tau - t)d\tau$. *Show that the samples of this waveform at multiples of* T *are* $R(mT) = S_m$, *for* $1 \leq m \leq K$.

(b) *Now suppose that the channel has an echo in it and behaves like a filter of impulse response* $f(t) = \delta(t) + \rho\delta(t - T))$, *where* ρ *is some constant between* -1 *and* 1. *Assume that the transmitted waveform* $S(t)$ *is filtered by* $f(t)$, *then filtered at the receiver by* $h(-t)$. *The resulting waveform* $\tilde{R}(t)$ *is again sampled at multiples of* T. *Determine the samples* $\tilde{R}(mT)$, *for* $1 \leq m \leq K$.

(c) *Suppose that the* k *th received sample is* $Y_k = S_k + \alpha S_{k-1} + Z_k$, *where* $Z_k \sim \mathcal{N}(0, \sigma^2)$ *and* $0 \leq \alpha < 1$ *is a constant.* S_k *and* S_{k-1} *are independent random variables that take on the values* 1 *and* -1 *with equal probability. Suppose that the detector decides* $\hat{S}_k = 1$ *if* $Y_k > 0$, *and decides* $\hat{S}_k = -1$ *otherwise. Find the probability of error for this receiver.*

Problem 4. (Matched Filter Intuition.)

In this problem, we develop some further intuition about matched filters.

We have seen that an optimal receiver front end for the signal set $\{s_j(t)\}_{j=0}^{m-1}$ reduces the received (noisy) signal $R(t)$ to the m real numbers $\langle R, s_j \rangle$, $j = 0, \ldots, m-1$. We gain additional intuition about the operation $\langle R, s_j \rangle$ by considering

$$R(t) \quad = \quad s(t) + N(t), \tag{3.3}$$

where $N(t)$ is additive white Gaussian noise of power spectral density $N_0/2$ and $s(t)$ is an arbitrary but fixed signal. Let $h(t)$ be an arbitrary waveform, and consider the receiver operation

$$Y \quad = \quad \langle R, h \rangle = \langle s, h \rangle + \langle N, h \rangle. \tag{3.4}$$

The signal-to-noise ratio (SNR) is thus

$$SNR \quad = \quad \frac{|\langle s, h \rangle|^2}{E\left[|\langle N, h \rangle|^2\right]}. \tag{3.5}$$

Notice that the SNR is not changed when $h(t)$ is multiplied by a constant. Therefore, we assume that $h(t)$ is a unit energy signal and denote it by $\phi(t)$. Then,

$$E\left[|\langle N, \phi \rangle|^2\right] \quad = \quad \frac{N_0}{2}. \tag{3.6}$$

(i) *Use Cauchy-Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy-Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $s(t)$?*

(ii) *To further illustrate this point, take ϕ and s to be two-dimensional vectors and use a picture to discuss why your result in* (i) *makes sense.*

(iii) *Take $\phi = (\phi_1, \phi_2)^T$ and $s = (s_1, s_2)^T$ and show how a high school student (without knowing about Cauchy-Schwarz inequality) would have found the matched filter.* Hint: *You have to maximize $\langle s, \phi \rangle$ subject to the constraint that ϕ has unit energy.*

(iv) *Hence to maximize the SNR, for each value of t we have to weigh (multiply) $R(t)$ with $s(t)$ and then integrate. Verify with a picture (convolution) that the output at time T of a filter with input $s(t)$ and impulse response $h(t) = s(T-t)$ is indeed $\int_0^T s^2(t)dt$.*

(v) *We may also look at the situation in terms of Fourier transforms. Write out the filter operation in the frequency domain. Express in terms of $S(f) = \mathcal{F}\{s(t)\}$.*

Problem 5. (Optimal receiver for signaling in non-white Gaussian noise.)

We consider the receiver design problem for signals used in non-white additive Gaussian noise. That is, we are given a set of signals $\{s_j(t)\}_{j=0}^{m-1}$ as usual, but the noise added to *those signals is no longer white; rather, it is a Gaussian stochastic process with a given power spectral density*

$$S_N(f) \;=\; G^2(f), \tag{3.7}$$

where we assume that $G(f) \neq 0$ *inside the bandwidth of the signal set* $\{s_j(t)\}_{j=0}^{m-1}$. *The problem is to design the receiver that minimizes the probability of error.*

(i) *Find a way to transform the above problem into one that you can solve, and derive the optimum receiver.*

(ii) *Suppose there is an interval* $[f_0, f_0 + \Delta]$ *inside the bandwidth of the signal set* $\{s_j(t)\}_{j=0}^{m-1}$ *for which* $G(f) = 0$. *What do you do? Describe in words.*

Problem 6. (Antipodal signaling in non-white Gaussian noise.)

In this problem, antipodal signaling (i.e. $s_0(t) = -s_1(t)$*) is to be used in non-white additive Gaussian noise of power spectral density*

$$S_N(f) \;=\; G^2(f), \tag{3.8}$$

where we assume that $G(f) \neq 0$ *inside the bandwidth of the signal* $s(t)$.

How should the signal $s(t)$ *be chosen (as a function of* $G(f)$*) such as to minimize the probability of error?* Hint: *For ML decoding of antipodal signaling in AWGN (of fixed variance), the* $Pr\{e\}$ *depends only on the signal energy.*

Problem 7. (Mismatched Receiver.)

Let the received waveform $Y(t)$ *be given by*

$$Y(t) \;=\; c\,X\,s(t) + N(t), \tag{3.9}$$

where $c > 0$ *is some deterministic constant,* X *is a random variable that takes on the values* $\{3, 1, -1, -3\}$ *equiprobably,* $s(t)$ *is the deterministic waveform*

$$s(t) \;=\; \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{otherwise,} \end{cases} \tag{3.10}$$

and $N(t)$ *is white Gaussian noise of spectral density* $\frac{N_0}{2}$.

(a) *Describe the receiver that, based on the received waveform* $Y(t)$, *decides on the value of* X *with least probability of error. Be sure to indicate precisely when your decision rule would declare "*$+3$*", "*$+1$*", "*-1*", and "*-3*".*

(b) *Find the probability of error of the detector you have found in Part* (a).

(c) *Suppose now that you still use the detector you have found in Part* (a)*, but that the received waveform is actually*

$$Y(t) = \frac{3}{4} c\, X\, s(t) + N(t), \tag{3.11}$$

i.e., you were mis-informed about the signal amplitude. What is the probability of error now?

(d) *Suppose now that you still use the detector you have found in Part* (a) *and that $Y(t)$ is according to Equation (3.9), but that the noise is* colored. *In fact, $N(t)$ is a zero-mean stationary Gaussian noise process of auto-covariance function*

$$K_N(\tau) = E[N(t)N(t+\tau)] = \frac{1}{4\alpha} e^{-|\tau|/\alpha}, \tag{3.12}$$

where $0 < \alpha < \infty$ is some deterministic real parameter. What is the probability of error now?

Problem 8. (QAM receiver)

Consider a transmitter which transmits waveforms of the form,

$$s(t) = \begin{cases} s_1\sqrt{\frac{2}{T}} \cos 2\pi f_c t + s_2\sqrt{\frac{2}{T}} \sin 2\pi f_c t, & \text{for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases} \tag{3.13}$$

where $2f_cT \in \mathbb{Z}$. $(s_1, s_2) \in \{(\sqrt{E}, \sqrt{E}), (-\sqrt{E}, \sqrt{E}), (-\sqrt{E}, -\sqrt{E}), (\sqrt{E}, -\sqrt{E})\}$ with equal probability. The signal received at the receiver is corrupted by AWGN of power spectral density $\frac{N_0}{2}$.

(a) *Specify the receiver for this transmission scheme.*
(b) *Draw the decoding regions and find the probability of error.*

Problem 9. *Consider the following functions $S_0(t)$, $S_1(t)$ and $S_2(t)$.*

(Gram-Schmidt for Three Signals)

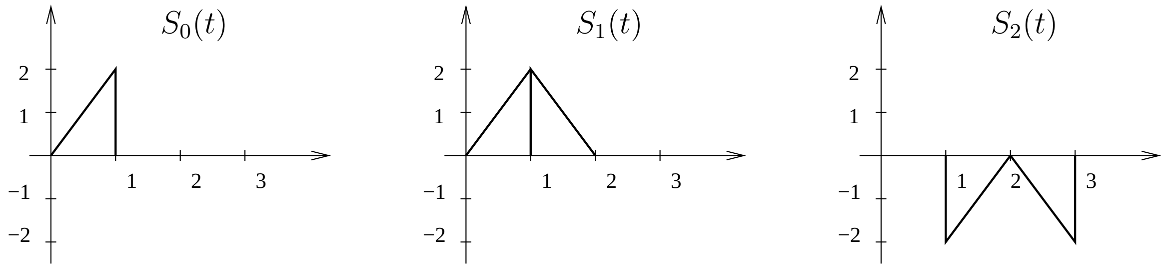
(i) *Using the Gram-Schmidt procedure, determine a basis of the space spanned by $\{s_0(t), s_1(t), s_2(t)\}$. Denote the basis functions by $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$.*

(ii) *Let*

$$V_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

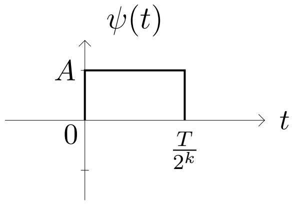
be two points in the space spanned by $\{\phi_0(t), \phi_1(t), \phi_2(t)\}$. What is their corresponding signal, $V_1(t)$ and $V_2(t)$? (You can simply draw a detailed graph.)

(iii) *Compute $\int V_1(t)V_2(t)dt$.*



PROBLEM 10. *Consider the following communication chain. We have 2^k possible hypotheses with $k \in \mathbb{N}$ to convey through a waveform channel. When hypothesis i is selected, the transmitted signal is $s_i(t)$ and the received signal is given by $R(t) = s_i(t) + N(t)$, where $N(t)$ denotes a white Gaussian noise with double-sided power spectral density $\frac{N_0}{2}$. Assume that the transmitter uses the position of a pulse $\psi(t)$ in an interval $[0, T]$, in order to convey the desired hypothesis, i.e., to send hypothesis i, the transmitter sends the signal $\psi_i(t) = \psi(t - \frac{iT}{2^k})$.*

(i) *If the pulse is given by the waveform $\psi(t)$ depicted below. What is the value of A that gives us signals of energy equal to one as a function of k and T?*



(ii) *We want to transmit the hypothesis $i = 3$ followed by the hypothesis $j = 2^k - 1$. Plot the waveform you will see at the output of the transmitter, using the pulse given in the previous question.*

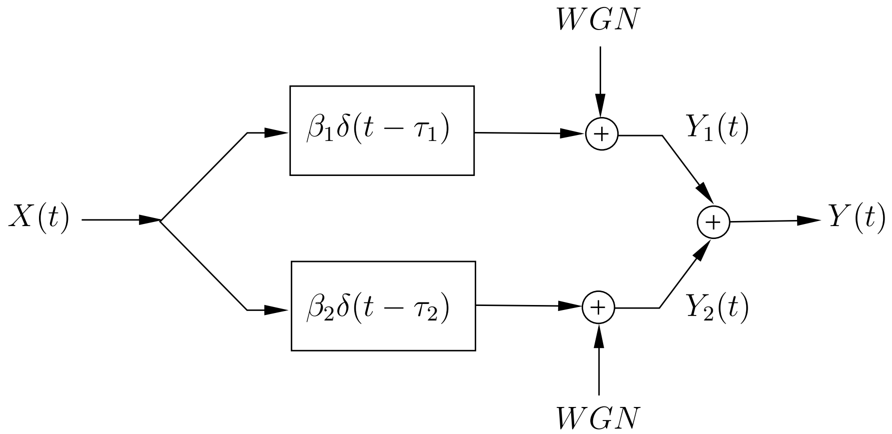
(iii) *Sketch the optimal receiver.*
What is the minimum number of filters you need for the optimal receiver? Explain.

(iv) *What is the major drawback of this signaling scheme? Explain.*

PROBLEM 11. (Communication Chain with two receive antennas)
Consider the following communication chain, where we have two possible hypotheses H_0 and H_1. Assume that $P_H(H_0) = P_H(H_1) = \frac{1}{2}$. The transmitter uses antipodal signaling. To transmit H_0, the transmitter sends a unit energy pulse $p(t)$, and to transmit H_1, it sends $-p(t)$. That is, the transmitted signal is $X(t) = \pm p(t)$. The observation consists of $Y_1(t)$ and $Y_2(t)$ as shown below. The signal along each "path" is an attenuated and delayed version of the transmitted signal $X(t)$. The noise is additive white Gaussian with double sided power spectral density $N_0/2$. Also, the noise added to the two observations

is independent and independent of the data. The goal of the receiver is to decide which hypothesis was transmitted, based on its observation.

We will look at two different scenarios: either the receiver has access to each individual signal $Y_1(t)$ and $Y_2(t)$, or the receiver has only access to the combined *observation $Y(t) = Y_1(t) + Y_2(t)$.*



a. *The case where the receiver has only access to the combined output $Y(t)$.*

1. *In this case, observe that we can write the received waveform as $\pm g(t) + Z(t)$. What are $g(t)$ and $Z(t)$ and what are the statistical properties of $Z(t)$?*

   Hint: Recall that $\int \delta(\tau - \tau_1)p(t-\tau)d\tau = p(t-\tau_1)$.

2. *What is the optimal receiver for this case? Your answer can be in the form of a block diagram that shows how to process $Y(t)$ or in the form of equations. In either case, specify how the decision is made between H_0 or H_1.*

3. *Assume that $\int p(t-\tau_1)p(t-\tau_2)dt = \gamma$, where $-1 \le \gamma \le 1$. Find the probability of error for this optimal receiver, express it in terms of the Q function, β_1, β_2, γ and $N_0/2$.*

b. *The case where the receiver has access to the individual observations $Y_1(t)$ and $Y_2(t)$.*

1. *Argue that the performance of the optimal receiver for this case can be no worse than that of the optimal receiver for part (a).*

2. *Compute the sufficient statistics (Y_1, Y_2), where $Y_1 = \int Y_1(t)p(t-\tau_1)dt$ and $Y_2 = \int Y_2(t)p(t-\tau_2)dt$. Show that this sufficient statistic (Y_1, Y_2) has the form $(Y_1, Y_2) = (\beta_1 + Z_1, \beta_2 + Z_2)$ under H_0, and $(-\beta_1 + Z_1, -\beta_2 + Z_2)$ under H_1, where Z_1 and Z_2 are independent zero-mean Gaussian random variables of variance $N_0/2$.*

3. *Using the LLR (Log-Likelihood Ratio), find the optimum decision rule for this case.*

Hint: *It may help to draw the two hypotheses as points in* $\mathbb{R}^2$. *If we let* $V = (V_1, V_2)$ *be a Gaussian random vector of mean* $m = (m_1, m_2)$ *and covariance matrix* $\Sigma = \sigma^2 I$, *then its pdf is* $p_V(v_1, v_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(v_1-m_1)^2}{2\sigma^2} - \frac{(v_2-m_2)^2}{2\sigma^2}\right)$.

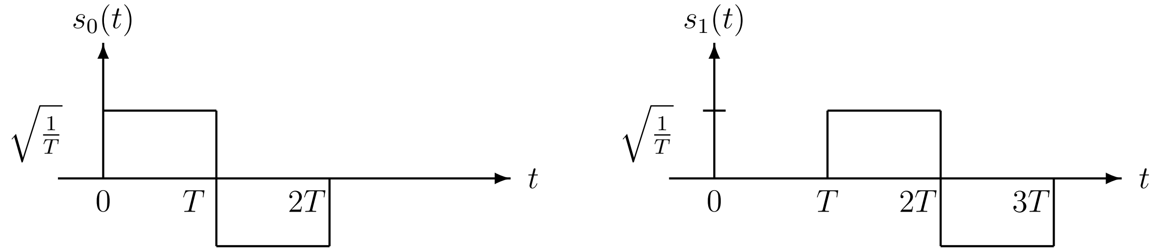
4. *What is the optimal receiver for this case? Your answer can be in the form of a block diagram that shows how to process* $Y_1(t)$ *and* $Y_2(t)$ *or in the form of equations. In either case, specify how the decision is made between* H_0 *or* H_1.
5. *Find the probability of error for this optimal receiver, express it in terms of the* Q *function,* β_1, β_2 *and* N_0.

c. *Comparison of the two cases*

1. *In the case of* $\beta_2 = 0$, *that is the second observation is solely noise, give the probability of error for both cases (a) and (b). What is the difference between them? Explain why.*

Problem 12. (Delayed Signals)

One of two signals shown in the figure below is transmitted over the additive white Gaussian noise channel. There is no bandwidth constraint and either signal is selected with probability $1/2$.



(a) *Draw a block diagram of a maximum likelihood receiver. Be as specific as you can. Try to use the smallest possible number of filters and/or correlators.*

(b) *Determine the error probability in terms of the* Q*-function, assuming that the power spectral density of the noise is* $\frac{N_0}{2} = 5 \ [\frac{W}{Hz}]$.

Appendix 3.A Rectangle and Sinc as Fourier Transform Pairs

The Fourier transform of a rectangular pulses is a sinc pulse. Often one has to go back and forth between such Fourier pairs. The purpose of this appendix is to make it easier to figure out the details.

First of all let us recall that a function g and its Fourier transform $g_{\mathcal{F}}$ are related by

$$g(u) = \int g_{\mathcal{F}}(\alpha) \exp(j2\pi u\alpha) d\alpha$$
$$g_{\mathcal{F}}(v) = \int g(\alpha) \exp(-j2\pi v\alpha) d\alpha.$$

Notice that $g_{\mathcal{F}}(0)$ is the area under g and $g(0)$ is the area under $g_{\mathcal{F}}$.

Next let us recall that $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ is the function that equals 1 at $x = 0$ and equals 0 at all other integer values of x. Hence if $a, b \in \mathbb{R}$ are arbitrary constants, $a\text{sinc}(bx)$ equals a at $x = 0$ and and equals 0 at nonzero multiples of $1/b$.

If you could remember that the area under $a\text{sinc}(bx)$ is a/b then, from the two facts above, you could conclude that its Fourier transform, which you know is a rectangle, has hight equals a/b and area a. Hence the width of this rectangle must be b.

It is actually easy to remember that the area under $a\text{sinc}(bx)$ is a/b: it is the area of the triangle described by the main lobe of $a\text{sinc}(bx)$, namely the area of the triangle with coordinates $(-1/b, 0)$, $(0, a)$, $(1/b, 0)$.

Appendix 3.B White Gaussian Noise

We assume that you are familiar with the concept of White Gaussian Noise. The purpose of this appendix is just to write down what you absolutely need to remember, for the purpose of this Chapter, about White Gaussian Noise.

If $N(t)$ is White Gaussian Noise of double-sided spectral density $\frac{N_0}{2}$ then:

- Its covariance function is
$$K_N(\tau) \triangleq \frac{N_0}{2}\delta(\tau), \ \forall\tau.$$
- Its spectrum (the Fourier transform of the covariance function) is
$$S_N(f) = \frac{N_0}{2}.$$
- If
$$Z_i = \int_{-\infty}^{\infty} N(t) g_i(t) dt, \quad i = 1, \ldots, K,$$

then $(Z_1, \ldots, Z_N)$ is a zero-mean Gaussian random vector and for any $1 \leq i,\ j \leq K$,

$$\begin{aligned} E[Z_i Z_j] &= E\left[\int_{-\infty}^{\infty} N(t) g_i(t) \int_{-\infty}^{\infty} N(\xi) g_i(\xi) d\xi\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[N(t)N(\xi)] g_i(H g_j(\xi) dt\xi \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N_0}{2} \delta(t-\xi) g_i(t) g_j(s) dt d\xi \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2} g_i(t) g_j(t) dt \end{aligned}$$

In particular, if $g_1(t), \ldots, g_k(t)$ are an orthonormal set then $Z_1, \ldots, Z_K$ are iid $\mathcal{N}(0, \frac{N_0}{2})$.