ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 16 — Information Theory and Coding
Midterm — December 16, 2005

Three problems with a total of 110 points.
Duration 2 hours and 45 minutes.
2 sheets of notes (4 pages) allowed.

PROBLEM 1. (30 points) Suppose that a source has alphabet $\mathcal{X}$, and, it is known that its distribution is either $p_1(x)$ or $p_2(x)$, …, or $p_K(x)$. Let $H_k = -\sum_x p_k(x) \log_2 p_k(x)$ denote the entropy of the distribution p_k, $k = 1, \dots, K$.

Define $\hat{p}(x) = \max_{1 \le k \le K} p_k(x)$, and $A = \sum_x \hat{p}(x)$.

(a) (10 pts) Show that $1 \le A \le K$.

(b) (10 pts) Show that there is a prefix free source code for X with codeword lenghts $l(x) = \lceil -\log_2 \hat{p}(x) + \log_2 A \rceil$.

(c) (10 pts) Show that, for a code as in (b), $\bar{L}_k = \sum_x p_k(x) l(x)$, (the average codeword length under distribution p_k), satisfies

$$H_k \le \bar{L}_k < H_k + \log_2 A + 1.$$

Problem 2. (40 points) In a casino one can bet on the outcome of a random variable X taking values in $\{1, \dots, K\}$. The distribution of X is $p(x)$. If $X = k$, the casino multiplies the money bet on outcome k by $\frac{1}{p(k)}$, the money bet on other outcomes are lost. The gambler's strategy is to set aside (that is, to not gamble with) a fraction $q(0)$ of his capital and to allocate the remaining capital among the different bets, betting a fraction $q(k)$ of his capital on outcome k. Thus, $\forall k, q(k) \geq 0$ and $\sum_{k=0}^{K} q(k) = 1$.

(a) (10 points) Suppose q is a strategy with $q(0) > 0$. Show that there exists a strategy $\hat{q}$ with $\hat{q}(0) = 0$ that performs identical to q, in the sense that the gambler will have an identical amount of money under any outcome of the random variable X for the two strategies.

From now on we will assume that $q(0) = 0$. Define

$$R_n = \frac{1}{n} \log \frac{C_n}{C_0}$$

to be the "rate of return" for the gambler where C_0 is the gambler's initial capital and C_n is the capital after playing n successive independent rounds of the game.

(b) (10 points) Use the law of large numbers to find

$$r = \lim_{n \to \infty} R_n$$

in terms of p, q.

(c) (10 points) Suppose that before every round i, the gambler is provided with a "side information" Y_i that is correlated with X_i (the distribution of (X_i, Y_i) being $p(x, y)$). The gambler's strategy in round i, given the side information $Y_i = y$, is to bet a fraction $q(k|y)$ of his capital on outcome k. Use the law of large numbers to recompute

$$r = \lim_{n \to \infty} R_n$$

in terms of p, q.

(d) (10 points) Find the strategy $q(x|y)$ that maximizes r and compare this maximum value to $I(X; Y)$.

Problem 3. (40 points) Let X be a random variable taking values in $\{0, 1\}$. Let Y be another random variable correlated with X taking values in $\mathcal{Y}$. The joint distribution of (X, Y) is $p_{XY}(x, y)$. Suppose the random variable Z is obtained from Y as $Z = g(Y)$ where $g(.)$ is a deterministic function.

(a) (10 points) Show that $I(X; Z) <= I(X; Y)$ with equality only if $p(xy|z) = p(x|z)p(y|z)$. [Hint: expand $I(X; Y, Z)$ using the chain rule.]

(b) (15 points) Suppose

$$g^*(y) = \log \frac{p(y|1)}{p(y|0)}$$

Show that with $Z^* = g^*(Y)$, we have $I(X; Z^*) = I(X; Y)$.

(c) (15 points) Suppose $g(.)$ is such that with $Z = g(Y)$ we have $I(X; Y) = I(X; Z)$. Show that one can recover Z^* from Z.