ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 18 Information Theory and Coding
Homework 8 November 20, 2007

PROBLEM 1. *Channels with memory have higher capacity.* Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$.

(a) Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Assume that $(Z_1, \ldots, Z_n)$ is independent of the input $(X_1, \ldots, X_n)$. Let $C = \log 2 - H(p, 1-p)$. Show that

$$\max_{p_{X_1,X_2,\ldots,X_n}} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC.$$

(b) Suppose that the $\{Z_i\}$ are generated as follows – $\Pr(Z_1 = 0) = \Pr(Z_1 = 1) = \frac{1}{2}$ and for $i \geq 1, \Pr(Z_{i+1} \neq Z_i) = q$.

(i) What is the marginal probability – $\Pr(Z_i = 1)$?

(ii) Justify the following sequence of steps:

$$\begin{aligned} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) &= H(Y_1, \ldots, Y_n) - H(Y_1, \ldots, Y_n | X_1, \ldots, X_n) \\ &\stackrel{a}{=} H(Y_1, \ldots, Y_n) - H(Z_1, \ldots, Z_n) \\ &\stackrel{b}{=} H(Y_1, \ldots, Y_n) - (H(Z_1) + \sum_1^n H(Z_{i+1}|Z_i)) \\ &= H(Y_1, \ldots, Y_n) - (1 + (n-1)h(q)) \\ &\stackrel{c}{\leq} (n-1)(1 - h(q)) \end{aligned}$$

Find the distribution on X^n, $p_{X_1,X_2,\ldots,X_n}$ that achieves the upper bound. This shows that the capacity of the channel with such a noise sequence is $1 - h(q)$.

PROBLEM 2. Consider two discrete memoryless channels. The input alphabet, output alphabet, transition probabities and capacity of the k'th channel is given by $\mathcal{X}_k$, $\mathcal{Y}_k$, p_k and C_k respectively ($k = 1, 2$). The channels operate independently. A communication system has access to both channels, that is, the effective channel between the transmitter and receiver has input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ and transition probabilities $p_1(y_1|x_1)p_2(y_2|x_2)$. Find the capacity of this channel.

PROBLEM 3. Show that a cascade of n identical binary symmetric channels,

$$X_0 \rightarrow \boxed{\text{BSC \#1}} \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow \boxed{\text{BSC \#n}} \rightarrow X_n$$

each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}\big(1 - (1-2p)^n\big)$ and hence that $\lim_{n\to\infty} I(X_0; X_n) = 0$ if $p \neq 0, 1$. Thus, if no processing is allowed at the intermediate terminals, the capacity of the cascade tends to zero.

PROBLEM 4. Consider a memoryless channel with transition probability matrix $P_{Y|X}(y|x)$, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For a distribution Q over $\mathcal{X}$, let $I(Q)$ denote the mutual information between the input and the output of the channel when the input distribution is Q. Show that for any two distributions Q and Q' over $\mathcal{X}$,

(a)

$$I(Q') \leq \sum_{x \in \mathcal{X}} Q'(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left(\frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

(b)

$$C \leq \max_{x} \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left(\frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

where C is the capacity of the channel. Notice that this upper bound to the capacity is independent of the maximizing distribution.