# Solutions to the Final Exam

## Problem 1

(a) Consider the convolution formula

$$v[n] = (x * y)[n] = \sum_k x[k]y[n-k].$$

To determine the support set of $v[n]$, we can find the values of $n$ for which the convolution is zero. In general, the right hand side is zero if and only if $n$ is such that for every $k \in \mathbb{Z}$, $x[k] = 0$ or $y[n-k] = 0$.

If $k \notin \{0, 1, \ldots, N-1\}$, $x[k] = 0$ since $x[n]$ has support of length $N$. If $k \in \{0, 1, \ldots, N-1\}$, then in general $x[k] \neq 0$, so we need $y[n-k] = 0$ in that case for the convolution to be zero. But since the support of $y[n]$ has length $M$, $y[n-k] = 0$ if $n < k$ or $n > M-1+k$, and since we want to know when this is true for all $0 \leq k \leq N-1$, we find that we need either $n < 0$ or $n > M+N-2$. Therefore, the support set of $v[n]$ has length $M+N-1$.

(b) From the properties of the DTFT, we know that convolution in time equals multiplication in frequency, so

$$V(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega}) = \left(\sum_n x[n]e^{-j\omega n}\right)\left(\sum_m y[m]e^{-j\omega m}\right).$$

(c) In Homework 3 and, in similar form, in the midterm exam, you have already learned that if the DTFT of a sequence of length $M+N-1$ is sampled at only $M$ points (assuming $M < M+N-1$), then the resulting sequence can be zero, even if the original sequence was non-zero. This should already make you suspect that the same is the case here: from (a) we know that $v[n]$ has length $M+N-1$, and we are sampling its DTFT at only $M$ points. This argumentation, while pointing in the right direction, is not sufficient to show that the claim can be true, however; we need to specifically show that it can be true when the sequence under consideration resulted from the convolution of two sequences of length $M$ and $N$, respectively.

Let us solve this problem for general values of $N$ and $M$. Using the multiplication property from (b) with $\omega = 2\pi k/M$, we have

$$\tilde{V}[k] = \left(\sum_n x[n]e^{-j\frac{2\pi kn}{M}}\right)\left(\sum_m y[m]e^{-j\frac{2\pi km}{M}}\right).$$

This expression is zero for all $k \in \{0, 1, \ldots, M-1\}$ if for each such $k$ at least one of the two terms on the right hand side is zero. We proceed now to construct $x[n]$ and $y[n]$ such that the first term is zero for $k = 0$, and the second term is zero for $k \in \{1, \ldots, M-1\}$.

For $k = 0$, the first term becomes $\sum_n x[n]$. Let us thus choose $x[n]$ such that this sum is zero, for instance by setting $x[n] = \delta[n] - \delta[n-1]$. (Note that a *non-zero* sequence $x[n]$ for which $\sum_n x[n] = 0$ exists only if $N > 1$.)

As for the second term, we can recognize it as the DFT of the (length-$M$) sequence $y[n]$. Since we want it to be zero for $k \in \{1, \ldots, M-1\}$, we can construct the DFT $Y[k]$ with this property, *e.g.*, $Y[k] = \delta[k]$. Then we take $y[n]$ to be the inverse DFT of $Y[k]$, *i.e.*, $y[n] = 1/M$ for $0 \leq n \leq M-1$.

In conclusion, provided that $1 < N \leq M$, we can always find two sequences $x[n]$ and $y[n]$ for which the property is true; in particular, this holds for the case where $N = 7$ and $M = 10$.

# Problem 2

(a) The important point you need to have in mind to solve this problem is the fact that for any sequence $t[n]$, we have $t[n] * \delta[n-m] = t[n-m]$.

Now, note that the sequence $v[n]$ shown in Fig. 1 is

$$v[n] = x[n] * u[n] - x[n] * u[n-2] = x[n] * (u[n] - u[n-1]) .$$

However, the transfer system from $x[n]$ to $v[n]$ can be simplified as

$$u[n] - u[n-2] = \delta[n] + \delta[n-1].$$

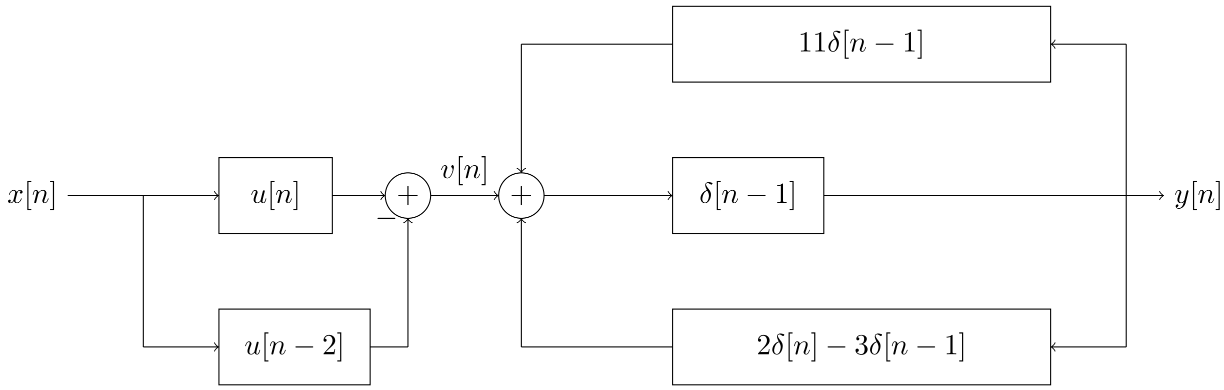


**Figure 1:** System with Impulse Response $h[n]$.

Therefore,

$$v[n] = x[n] * (\delta[n] + \delta[n-1]) = x[n] + x[n-1].$$

In the second part of the system with feedback, we have

$$\begin{aligned} y[n] &= (y[n] * 11\delta[n-1] + y[n] * (2\delta[n] - 3\delta[n-1]) + v[n]) * \delta[n-1] \\ &= (11y[n-1] + 2y[n] - 3y[n-1] + x[n] + x[n-1]) * \delta[n-1] \\ &= 8y[n-2] + 2y[n-1] + x[n-1] + x[n-2], \end{aligned}$$

which is the difference equation of the system.

(b) You can simply find the system transfer function by taking the $z$-transform of the difference equation.

$$Y(z) = 8z^{-2}Y(z) + 2z^{-1}Y(z) + z^{-1}X(z) + z^{-2}X(z)$$

Therefore,

$$Y(z)(1 - 2z^{-1} - 8z^{-2}) = X(z)(z^{-1} + z^{-2})$$

and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} - 8z^{-2}}.$$

(c) The zeros and poles of a transfer function are the roots of its nominator and denominator, respectively.

$$\begin{array}{llllll} \text{zeros:} & z^{-1} + z^{-2} = 0 & \Rightarrow & z^{-1} = -1, 0 & \Rightarrow & z = -1, \infty \\ \text{poles:} & 1 - 2z^{-1} - 8z^{-2} = 0 & \Rightarrow & z^{-1} = -\frac{1}{2}, \frac{1}{4} & \Rightarrow & z = -2, 4. \end{array}$$

None of the zeros and poles are repeated, and the multiplicity of all is 1.

(d) Having the assumption $y[n] = 0$ for $n < 0$, we conclude that the system is causal. Among all possible regions of convergence (characterized by the poles of $H(z)$), $|z| > 4$ is the only one corresponding to a right-sided sequence. By doing partial fraction of $H(z)$, we have

$$H(z) = \frac{z^{-1}(1 + z^{-1})}{(1 + 2z^{-1})(1 - 4z^{-1})} = z^{-1}\left(\frac{A}{1 + 2z^{-1}} + \frac{B}{1 - 4z^{-1}},\right)$$

where $A$ and $B$ have to computed such that the equality holds. By solving a system of linear equations, we can find them as

$$\begin{cases} A + B = 1 \\ -4A + 2B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{6} \\ B = \frac{5}{6} \end{cases}$$

Hence,

$$H(z) = z^{-1}\left(\frac{1}{6}\frac{1}{1 + 2z^{-1}} + \frac{5}{6}\frac{1}{1 - 4z^{-1}}\right),$$

and

$$\begin{aligned} h[n] &= \delta[n-1] * \left(\frac{1}{6}(-2)^n u[n] + \frac{5}{6}4^n u[n]\right) \\ &= \frac{1}{6}(-2)^{n-1}u[n-1] + \frac{5}{6}4^{n-1}u[n-1] \\ &= \frac{1}{6}(-2)^{n-1}u[n-1] + \frac{5}{6}4^{n-1}u[n-1] \\ &= \left(-\frac{1}{12}(-2)^n + \frac{5}{24}4^n\right)u[n] - \frac{1}{8}\delta[n]. \end{aligned}$$

(e) It is clear from the transfer function that the system is linear.

(f) Since all the filters used in the system are time-invariant, the whole system is also time-invariant.

(g) Since the region of convergence is $|z| > 4$, and it does not contain the unit circle, the DTFT is not defined and system is not stable.

## Problem 3

(a) From the sampling theorem, we know that $\Omega_s = 2\Omega_N$ is such that any sampling frequency larger than $\Omega_s$ allows perfect reconstruction of $x_c(t)$.

(b) Yes, perfect reconstruction is possible, because $\Omega_s' \geq \Omega_s$.

(c) The DTFT of the sample sequence $u[n]$ is given in Figure 2.

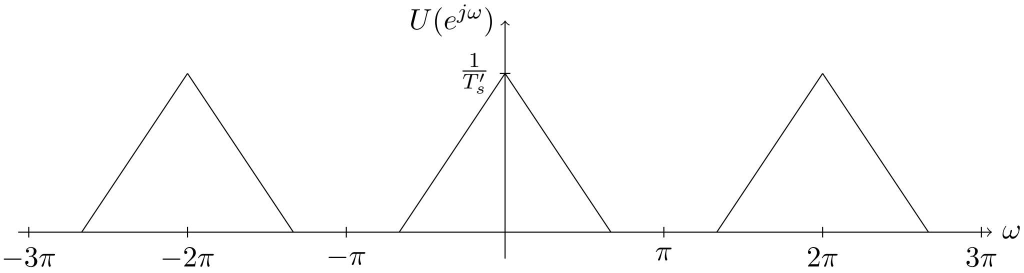


**Figure 2:** Problem 3 (c)

(d) Upsampling by a factor of 2 has the effect of "contracting" the DTFT horizontally by a factor of 2. Hence, we obtain the DTFT in Figure 3.

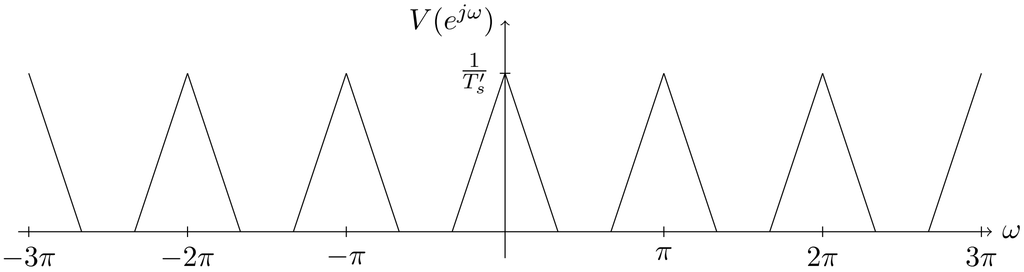


**Figure 3:** Problem 3 (d)

(e) Consider first the continuous-time function

$$y_s(t) = \sum_{n=-\infty}^{\infty} v[n]\delta(t - nT_s').$$

This is just a sequence of delta-pulses, whose amplitudes are $v[0], v[1], v[2], \ldots$ (we call this a pulse-train). The CTFT of $y_s(t)$ is given in Figure 4. The sinc interpolation of $v[n]$ is

$$y_c(t) = \sum_{n=-\infty}^{\infty} v[n] \operatorname{sinc}\left(\frac{t - nT_s'}{T_s'}\right).$$

Hence, it can be shown that $y_c(t) = y_s(t) * \operatorname{sinc}(\frac{t}{T_s'})$. In the frequency domain, this convolution corresponds to the multiplication

$$Y_c(j\Omega) = Y_s(j\Omega) T_s' \mathbf{1}_{[-\frac{\Omega_s'}{2}, \frac{\Omega_s'}{2}]},$$

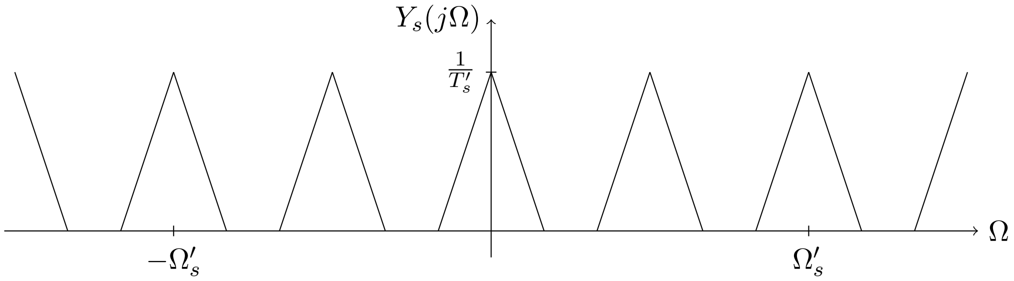


**Figure 4:** Problem 3 (e)

where the box function $T'_s \mathbf{1}_{[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}]}$ is simply the CTFT of $\text{sinc}(\frac{t}{T'_s})$. The resulting CTFT $Y_c(j\Omega)$ is given in Figure 5.

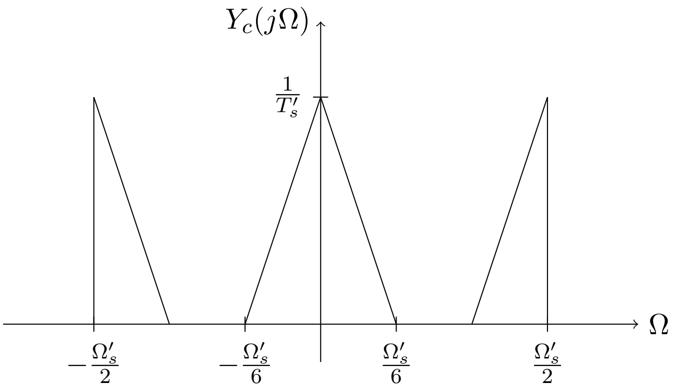


**Figure 5:** Problem 3 (e)

(f) Here, the spectrum $Y_c(j\Omega)$ has support $[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}]$. According to the sampling theorem, the limit sampling frequency is equal to the width of this support, namely $\tilde{\Omega}_s = 2\frac{\Omega'_s}{2} = \Omega'_s$.

(g) To find out how to sketch the DTFT of $w[n]$, we observe that the support of $Y_c(j\Omega)$ is $[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}] = [-\frac{3}{2}\Omega_N, \frac{3}{2}\Omega_N]$. Now, we sample $y_c(t)$ at $\hat{\Omega}_s = 2\Omega_N$, which will correspond to $2\pi$ in the DTFT sketch. Hence, the support of one "copy" of the spectrum in the DTFT is $[-\frac{3}{2}\pi, \frac{3}{2}\pi]$. The result is shown in Figure 6.

(h) Yes, perfect reconstruction is possible. We can see this because there is no aliasing in the DTFT in Figure 6. First, we sketch the CTFT of $\hat{y}_s(t)$, which is the pulse-train that corresponds to $w[n]$. This sketch is shown in Figure 7.

To recover $y_c(t)$, we can use a filter that combines a low-pass and a band-pass filter. Such a filter $H_r(j\Omega)$ is shown in Figure 8.

Mathematically, the frequency response of the filter is

$$H_r(j\Omega) = \begin{cases} \hat{T}_s & \text{if } |\Omega| \in [\frac{1}{2}\hat{\Omega}_s, \frac{3}{4}\hat{\Omega}_s], \\ \hat{T}_s & \text{if } |\Omega| \in [0, \frac{1}{4}\hat{\Omega}_s], \\ 0 & \text{otherwise.} \end{cases}$$

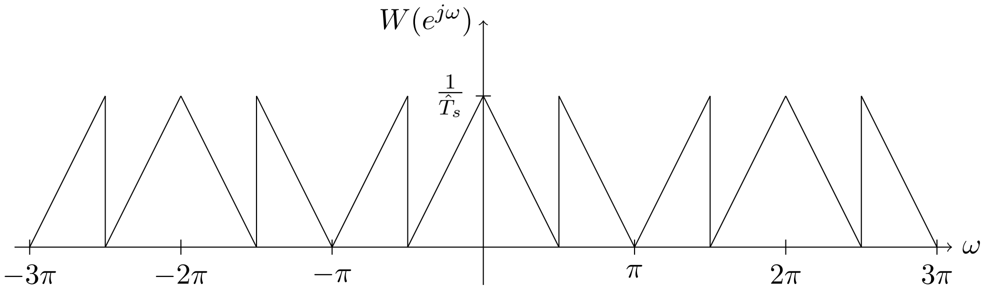


**Figure 6:** Problem 3 (g)

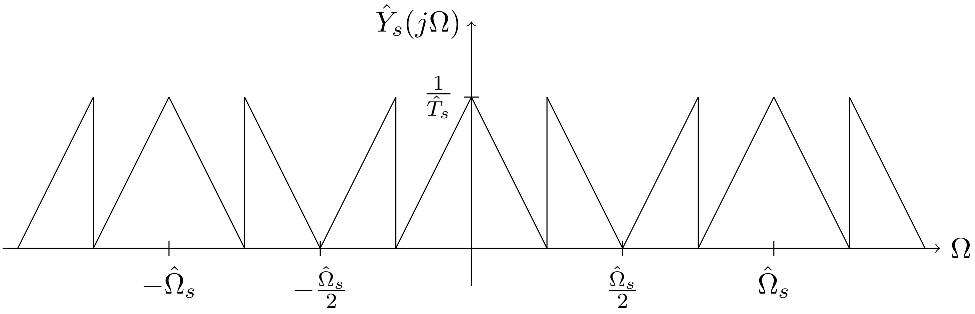


**Figure 7:** Problem 3 (h)

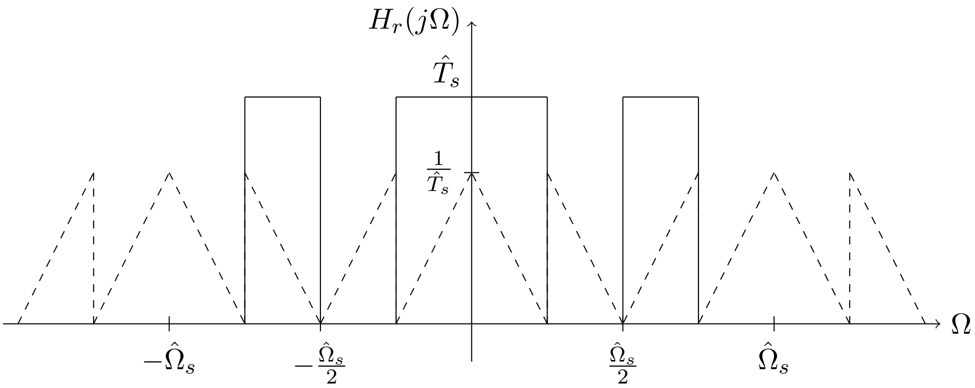


**Figure 8:** Problem 3 (h)

## Problem 4

(a) Since the system from $x[n]$ and $y[n]$ to $u[n]$ is LTI, we can simply write

$$u[n] = U_2(x[n]) + U_2(y[n]) * \delta[n-1].$$

So, is it clear that

$$u[n] = \begin{cases} x[\frac{n}{2}] & \text{if } n \text{ is even} \\ y[\frac{n-1}{2}] & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, after the time-variant operator, $\mathcal{S}$, we get

$$v[n] = \mathcal{S}\{u[n]\} = \begin{cases} x[k] & \text{if } n = 2k \\ \frac{1}{2}x[k] + \frac{1}{2}y[k] & \text{if } n = 2k+1. \end{cases} \tag{1}$$

(b) The system shown in Fig. 9 is equivalent to the system considered in the problem. Note that all the filters used in the equivalent system are linear and time-invariant.

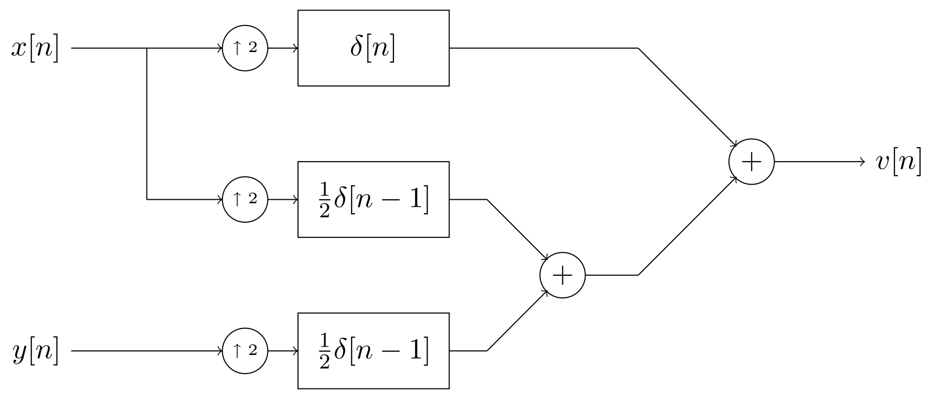


**Figure 9:** Equivalent multirate system

(c) It is clear from the equivalent system that

$$v[n] = U_2(x[n]) * \delta[n] + U_2(x[n]) * \frac{1}{2}\delta[n-1] + U_2(y[n]) * \frac{1}{2}\delta[n-1].$$

Therefore,

$$\begin{aligned} V(z) &= X(z^2) + \frac{1}{2}z^{-1}X(z^2) + \frac{1}{2}z^{-1}Y(z^2) \\ &= (1 + \frac{1}{2}z^{-1})X(z) + \frac{1}{2}z^{-1}Y(z). \end{aligned}$$

(d) $a[n]$ is just the down-sampled version of $v[n]$ by a factor 2, *i.e.*, it only contains the even-index samples of $v[n]$. So, from (1), it is clear that

$$a[n] = v[2n] = x[n].$$

We can can also easily see that $b[n]$ takes the odd-indices of the input sequence. Therefore,

$$b[n] = D_2(v[n] * \delta[n+1]) = D_2(v[n+1]) = v[2n+1] = \frac{1}{2}x[n] + \frac{1}{2}y[n].$$

(e) Having the system in part (d), it is easy to do a little modification to come up with the inverse system. Note that $x[n]$ is already produced at one of the output branches, and we only have to scale the other output, and subtract $x[n]$. Hence, we obtain the following system.

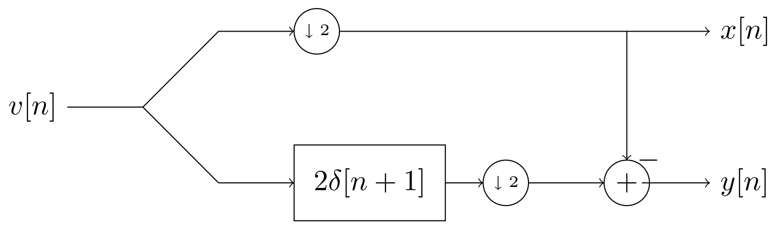


**Figure 10:** Inverse system

# Problem 5

(a) Using the Noble identities, we can transform the given filter bank as shown in Figure 11. The resulting filters are

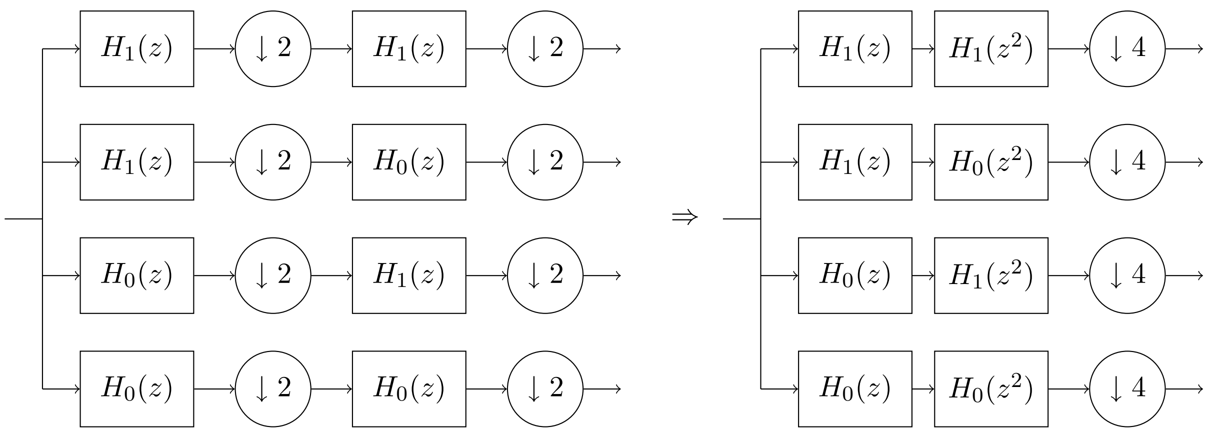


**Figure 11:** Transformation of the filter bank of Problem 5(a) using the Noble identities.

$$
\begin{aligned}
\tilde{h}^{(1)}[n] &= \frac{1}{2}(\delta[n] - \delta[n+1] - \delta[n+2] + \delta[n+3]) \\
\tilde{h}^{(2)}[n] &= \frac{1}{2}(\delta[n] - \delta[n+1] + \delta[n+2] - \delta[n+3]) \\
\tilde{h}^{(3)}[n] &= \frac{1}{2}(\delta[n] + \delta[n+1] - \delta[n+2] - \delta[n+3]) \\
\tilde{h}^{(4)}[n] &= \frac{1}{2}(\delta[n] + \delta[n+1] + \delta[n+2] + \delta[n+3]).
\end{aligned}
$$

This can be seen either visually, computing directly the convolution of the respective filters, or by going through the Z-transform, which, for instance, gives for the first branch

$$H^{(1)}(z) = \frac{1}{2}(1-z)(1-z^2) = \frac{1}{2}(1 - z - z^2 + z^3).$$

$h^{(1)}[n]$ then follows easily from the correspondence $z^i \leftrightarrow \delta[n+i]$.

(b) We can write the output of branch $i$ as

$$v_i = (x * h^{(i)})[0] = \sum_{k=0}^{3} x[k]h^{(i)}[-k],$$

from which we find $M_{(i,k)} = h^{(i)}[-k]$. The resulting matrix is then

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

(c) Let us study the matrix $\mathbf{M}'$. Our first observation is that the first two rows of $\mathbf{M}'$ are the same as rows 4 and 3 of the matrix $\mathbf{M}$ of Question (b). We can therefore simply take the corresponding branches of the original filter tree and install them into our new filter tree. Next, consider the last two rows of $\mathbf{M}'$. The corresponding filter bank outputs are $v'_3 = x[0] - x[1]$ and $v'_4 = x[2] - x[3]$. This is as if first $x[0], x[1]$ and then $x[2], x[3]$ were passed through the filter $h_1[n]$. Thus, the last two rows of $\mathbf{M}'$ can be implemented by a single branch consisting of the single filter $h_1[n]$ followed by downsampling by a factor 2. Figure 12 shows the final filter bank tree.

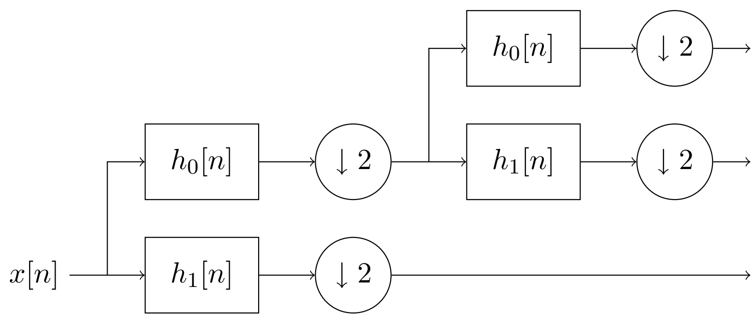


**Figure 12:** Filter tree corresponding to the matrix $\mathbf{M}'$ of Problem 5(c). Every time the first two branches produce one output each, the third branch produces two outputs.