FINAL

Wednesday March 7, 2007, 14:15-18:15

This exam has 5 problems and 100 points in total .

Instructions

- You are allowed to use 2 sheets of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- It is your responsibility to number the pages of your solutions and write on the first sheet the total number of pages submitted.

Good Luck!

Problem 1

[**(17 pts)**]

Suppose that $x[n]$ is a *non-zero* sequence of length N *i.e,*

$$x[n] = 0 \quad , \text{for } n < 0 \text{ and } n > N-1$$

and $x[n] \neq 0$ for some $n \in 0 \leq n \leq N-1$. Further, let us define

$$X[k] = \sum_n x[n] e^{-j\frac{2\pi}{M}kn} \quad , \text{for } k = 0, ..., M-1$$

(a) Prove or disprove the following:

[6pts] (i) It is possible that $X[k] = 0$ for *all* $k = 0, ..., M-1$ if $M \geq N$. If so, give an example. If not, prove that it is not possible.

[6pts] (ii) It is possible that $X[k] = 0$ for *all* $k = 0, ..., M-1$ if $M < N$. If so, give an example. If not, prove that it is not possible.

[5pts] (b) Now, set $M = 2N$ and define:

$$X_1[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \quad , \text{for } k = 0, ..., N-1$$
$$X_2[k] = \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}kn} \quad , \text{for } k = 0, ..., M-1$$

What is the relationship between $X_1[k]$ and $X_2[k]$?

Problem 2

[**(14 pts)**]

Suppose that we know that:

$$w[n] = \begin{cases} \frac{1}{n+1} h[n] & , \text{for } n > 0 \\ 0 & , \text{else} \end{cases}$$

[*3pts*] (a) If $h[n]$ is a strictly causal sequence (*i.e.*, $h[0] = 0,\ n \leq 0$) then find $H(z)$ in terms of $W(z)$ and the corresponding ROC $\mathcal{R}_h$ in terms of $\mathcal{R}_w$, the ROC of $W(z)$.

[*2pts*] (b) If $w[n] = a^n u[n-1]$, find $W(z)$, the z-transform of $w[n]$ and its corresponding ROC $\mathcal{R}_w$.

[*2pts*] [*2pts*] (c) Find $h[n]$ corresponding to the $w[n]$ given in part (b). Does this correspond to a stable system? Note that your answer can depend on a.

[*2pts*] (d) Suppose that $G(z)$ is known to be a system such that the relationship shown in Fig. 1 holds. Express $G(z)$ in terms of $H(z)$.

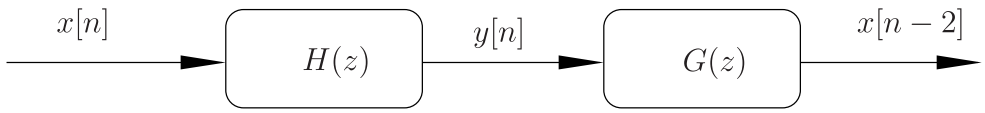

Figure 1: Relationship for problem 2(d)

[*5pts*] (e) Given the $H(z)$ and the corresponding ROC found in (c), state if $G(z)$ can be causal and can it be stable.

Problem 3

[**(19 pts)**]

We are considering a continuous-time signal $x_c(t)$ with corresponding continuous-time Fourier transform $X_c(j\Omega)$ given in Figure 2.

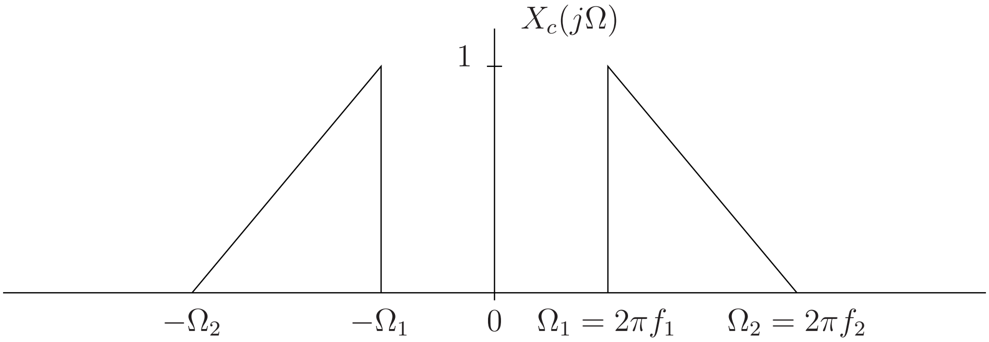

Figure 2: Continuous-time Fourier transform of $x_c(t)$ in Problem 3.

[*2pts*] (a) Suppose that we are sampling $x_c(t)$ with period T_s, i.e. we create

$$x[n] = x_c(nT_s), \qquad n \in \mathbb{Z}.$$

According to the sampling theorem, what is the maximum sampling period T_s for which $x_c(t)$ is recoverable from $\{x[n]\}$?

[*1pts*] (b) Suppose $\Omega_1 = 2\pi \cdot 150$, $\Omega_2 = 2\pi \cdot 200$. Let us sample at rate $f'_s = 100$ Hz. Does this satisfy the condition given in part (a)?

[*5pts*] (c) Let $v[n] = x_c(nT'_s)$, $n \in \mathbb{Z}$, with $f'_s = \frac{1}{T'_s} = 100$ Hz. Sketch the spectrum of $v[n]$.

[*4pts*] (d) Let $x_s(t) = \sum_n v[n]\delta(t - nT'_s)$ be a continuous-time signal. Sketch the continuous-time Fourier transform $X_s(j\Omega)$ of $x_s(t)$.

[*7pts*] (e) Can we recover $x_c(t)$ from $v[n]$? If so, clearly and explicitly demonstrate the method. If not, explain.
Hint: Can $x_c(t)$ be reconstructed from $x_s(t)$ found in part (d) of problem?

Problem 4

[(**25 pts**)]

We consider the two systems given in Figure 3.

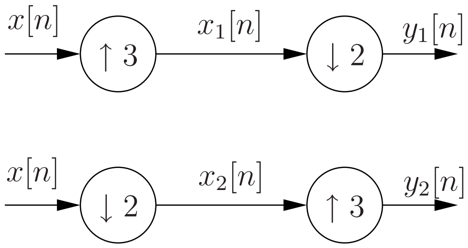

Figure 3: The two systems considered in Problem 4 .

[*5pts*] (a) For $X(e^{j\omega})$ as given in Figure 4, sketch $X_1(e^{j\omega})$ and $Y_1(e^{j\omega})$.

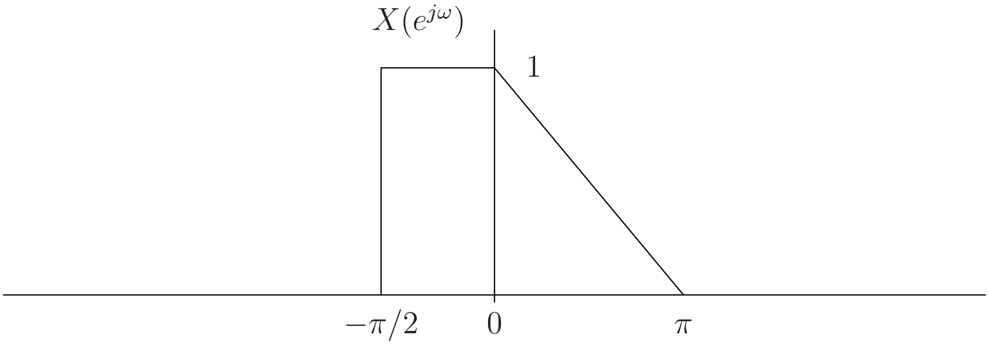

Figure 4: $X(e^{j\omega})$ for Problem 4 .

[*5pts*] (b) Again for $X(e^{j\omega})$ as given in Figure 4, sketch $X_2(e^{j\omega})$ and $Y_2(e^{j\omega})$.

[*1pts*] (c) What is the relationship between $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$?

For parts $(d), (e)$ we consider a general spectrum $X(e^{j\omega})$ and *not* specific to Figure 4. Therefore, the answers would be in terms of a general spectrum $X(e^{j\omega})$.

[*10pts*] (d) For general $X(e^{j\omega})$, write both $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$ in terms of $X(e^{j\omega})$. (*Hint: Use the properties of the Fourier transform of upsampled and downsampled signals as done in class.*)

[*4pts*] (e) Using the expressions you have obtained in part (d) prove the relationship between $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$.
Hint: the relationship discovered in part (c) should be generalized to an arbitrary $X(e^{j\omega})$

Problem 5

[(**25 pts**)]

Consider the analysis filterbank given in Figure 5.

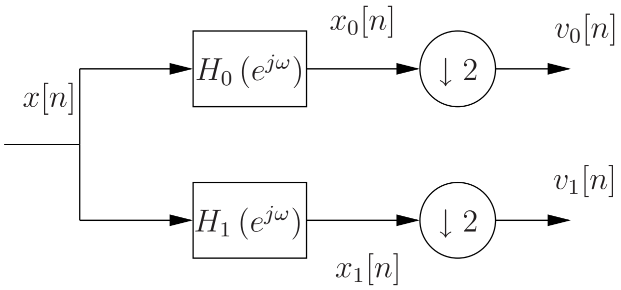

Figure 5: Analysis filterbank for Problem 5.

Let $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ be as given in Figure 6.

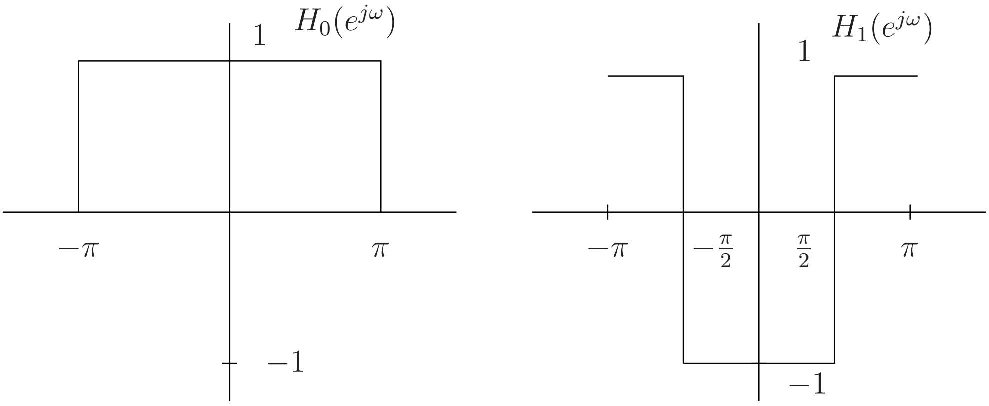

Figure 6: $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ for Problem 5 .

Let $X(e^{j\omega})$ be as given in Figure 7.

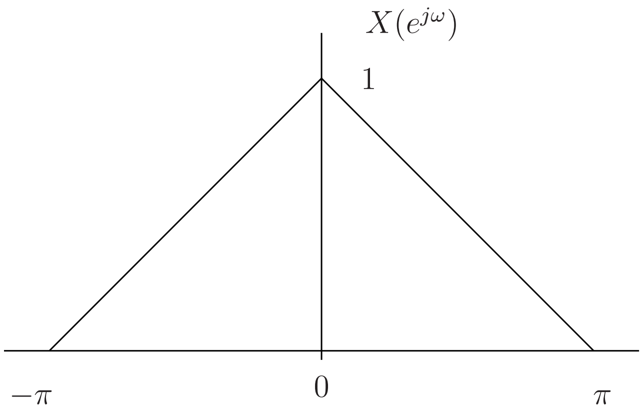

Figure 7: $X(e^{j\omega})$ for Problem 5.

[*6pts*] (a) Sketch $X_0(e^{j\omega})$, $X_1(e^{j\omega})$, $V_0(e^{j\omega})$ and $V_1(e^{j\omega})$.

Figure 8 shows the corresponding synthesis filterbank. In the remaining part of the exercise you are asked to find $F_0(e^{j\omega})$ and $F_1(e^{j\omega})$ such that we have perfect reconstruction, i.e. $\hat{X}(e^{j\omega}) = X(e^{j\omega})$.

[*3pts*] (b) Sketch $Y_0(e^{j\omega})$, and $Y_1(e^{j\omega})$.

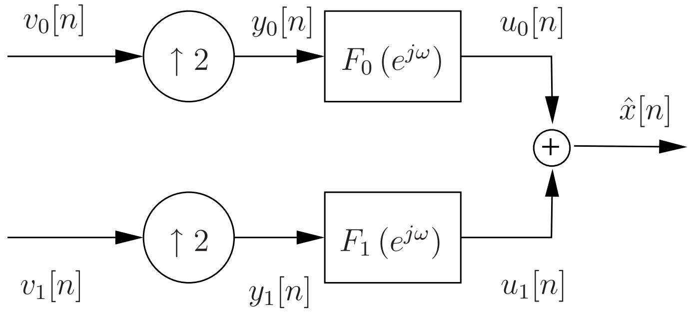

Figure 8: Synthesis filter bank for Problem 5.

[*8pts*] (c) Find $F_0\left(e^{j\omega}\right)$ and $F_1\left(e^{j\omega}\right)$ for which we have perfect reconstruction. For this choice, sketch $U_0\left(e^{j\omega}\right)$ and $U_1\left(e^{j\omega}\right)$.*(Hint: See Figure 9 for choices for $F_0\left(e^{j\omega}\right)$ and $F_1\left(e^{j\omega}\right)$. You can choose $F_0\left(e^{j\omega}\right)$ and $F_1\left(e^{j\omega}\right)$ from among these choices.).*

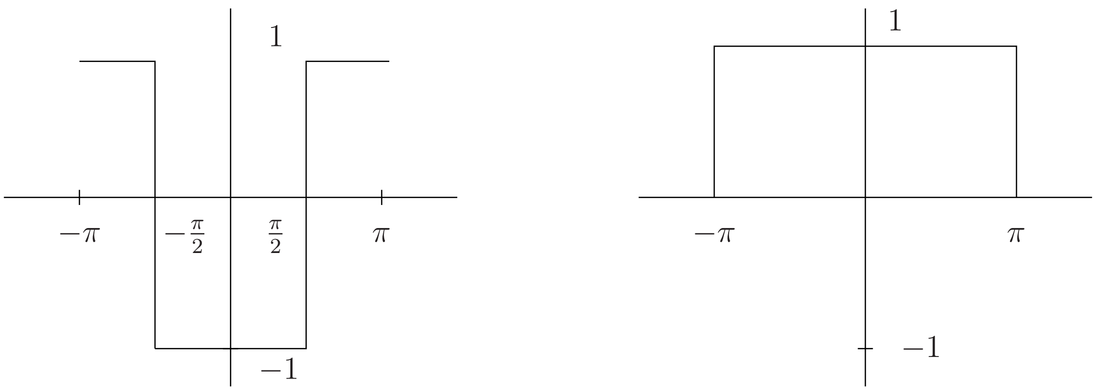

Figure 9: Hints for choices of $F_0\left(e^{j\omega}\right)$ and $F_1\left(e^{j\omega}\right)$ for Problem 5 .

For part (d) of this problem, we use *general* spectra $X(z)$ and filters $H_0(z), H_1(z), F_0(z), F_1(z)$.

[*8pts*] (d) Write an expressions for $\hat{X}(z)$, the Z-transform of $\hat{x}[n]$ as

$$\hat{X}(z) = T(z)X(z) + A(z)X(-z),$$

i.e., explicitly find $T(z), A(z)$ in terms of $H_0(z), H_1(z), F_0(z), F_1(z)$.

Given this, write the conditions for perfect reconstruction, *i.e.*, for $\hat{x}[n] = x[n]$?