Addendum to Chapter 11: Orthogonal Filter Banks

1 Two-Channel Orthogonal Expansions and Perfect Reconstruction

The Haar and sinc expansions of discrete-time signals that we have seen earlier are both orthonormal expansions that can be implemented by perfect reconstruction filter banks. The goal of the present section is to show that these two examples are not the only expansions with this property, but that for a wide class of orthonormal expansions we can always find a filter bank that implements the expansion.

Let $\{\varphi_k[n]\}_{k\in\mathbb{Z}}$ be a basis of the form

$$\begin{aligned}\varphi_{2k}[n] &= \varphi_0[n-2k]\\ \varphi_{2k+1}[n] &= \varphi_1[n-2k]\end{aligned}$$

whose elements satisfy the orthonormality conditions

$$\langle \varphi_i[n-2k], \varphi_j[n-2l] \rangle = \delta[i-j]\delta[k-l], \quad i,j \in \{0,1\}. \tag{1}$$

Consider now the filter bank of Figure 1 with analysis and synthesis filters

$$g_i[n] = \varphi_i[n] \quad \text{and} \quad h_i[n] = g_i[-n], \quad i = 0,1. \tag{2}$$

The goal of this section is to show that if (1) and (2) hold, then this filter bank has perfect reconstruction, i.e., $\hat{x}[n] = x[n]$.

As we have previously seen, the signals at the output of the filter bank's synthesis filters are

$$U_i(z) = \frac{1}{2} G_i(z) \left[H_i(z)X(z) + H_i(-z)X(-z) \right], \quad i = 0,1. \tag{3}$$

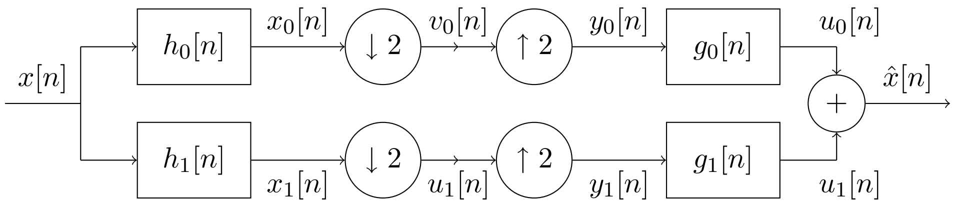

Figure 1: A General two-channel orthogonal FIR filter bank.

Since $h_i[n] = g_i[-n]$, $H_i(z) = G_i(z^{-1})$. Using this together with (3) we can write the reconstructed signal $\hat{X}(z) = U_0(z) + U_1(z)$ in matrix form as

$$\hat{X}(z) = \frac{1}{2} \begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} G_0(z^{-1}) & G_0(-z^{-1}) \\ G_1(z^{-1}) & G_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}.$$

The reconstruction is perfect ($\hat{X}(z) = X(z)$ for all $X(z)$) if and only if

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} G_0(z^{-1}) & G_0(-z^{-1}) \\ G_1(z^{-1}) & G_1(-z^{-1}) \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}. \tag{4}$$

We will now show that this last equality indeed follows from properties (1) and (2).

Let us start by defining the *autocorrelation sequence* $p_i[n]$ of $g_i[n]$ as

$$p_i[l] = \langle g_i[n], g_i[n-l] \rangle, \quad i = 0, 1. \tag{5}$$

Then $q_i[n] = p_i[2n]$, the downsampled version of the autocorrelation sequence, satisfies

$$q_i[n] = \delta[n]. \tag{6}$$

This follows from the orthonormality property (1) of $g_i[n]$ and from the definition of $p_i[n]$. Note now that that another way of writing (5) is $p_i[n] = g_i[n] * g_i[-n]$, or, after taking the Z-transform, $P_i(z) = G_i(z)G_i(z^{-1})$. The Z-transform of $q_i[n]$ is then, making use of (6),

$$Q_i(z) = \frac{1}{2} \left[G_i(z^{1/2})G_i(z^{-1/2}) + G_i(-z^{1/2})G_i(-z^{-1/2}) \right] = 1,$$

or equivalently

$$2Q_i(z^2) = G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2, \quad i = 0, 1, \tag{7}$$

where we have used the downsampling property of the Z-transform. In a similar way we can define the *crosscorrelation sequence* $p_{01}[n]$ of $g_0[n]$ and $g_1[n]$ as

$$p_{01}[l] = \langle g_0[n], g_1[n-l] \rangle,$$

and let $q_{01}[n] = p_{01}[2n]$. Then we have

$$q_{01}[n] = 0 \quad \text{for all } n,$$

where this last equality follows again from the orthonormality property (1). Proceeding in a similar way as for $q_i[n]$ above, we obtain

$$2Q_{01}(z^2) = G_0(z)G_1(z^{-1}) + G_0(-z)G_1(-z^{-1}) = 0. \tag{8}$$

We can now combine (7) and (8) in matrix form to get

$$\underbrace{\begin{bmatrix} G_0(z^{-1}) & G_0(-z^{-1}) \\ G_1(z^{-1}) & G_1(-z^{-1}) \end{bmatrix}}_{\mathbf{G}_m^T(z^{-1})} \underbrace{\begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}}_{\mathbf{G}_m(z)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{9}$$

The first row of $\mathbf{G}_m(z)$ can be written in the following somewhat complicated form:

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{1}{2}\mathbf{G}_m(z). \tag{10}$$

Since, from (9), $(\mathbf{G}_m^T(z^{-1}))^{-1} = (1/2)\mathbf{G}_m(z)$, we can multiply both sides of (10) on the right by $\mathbf{G}_m^T(z^{-1})$ to finally obtain

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \mathbf{G}_m^T(z^{-1}) = \begin{bmatrix} 2 & 0 \end{bmatrix},$$

which is precisely the condition for perfect reconstruction found in (4).

2 Tree Structured Filter Banks and Wavelet Transform

Tree structured filter banks can be constructed using a cascade of two-channel filter banks as shown in Figures 2 and 3. The structure given in Figures 2 and 3 is called a *constant-Q* or *constant relative bandwidth filter bank* since the bandwidth of each channel divided by its center frequency is a constant. This type of cascaded filter bank is also called an *octave-band filter bank*.

If the two-channel filter bank used in a tree structured filter bank is orthonormal, then it implements what is called an orthonormal *discrete-time wavelet series*, or sometimes *discrete-time wavelet transform*.

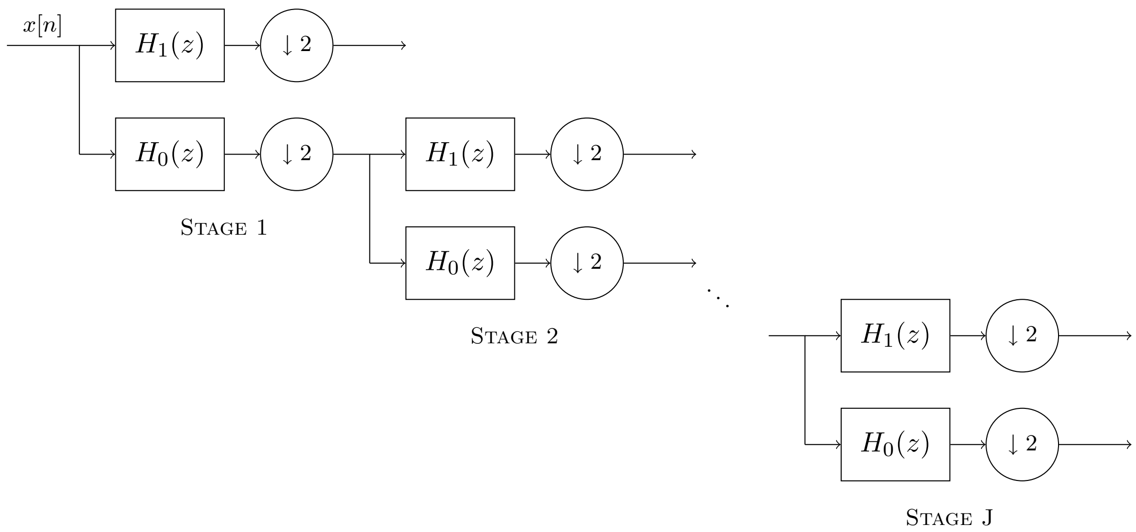

Figure 2: The analysis part of a tree-structured filter bank consisting of a cascade of J two-channel filter banks.

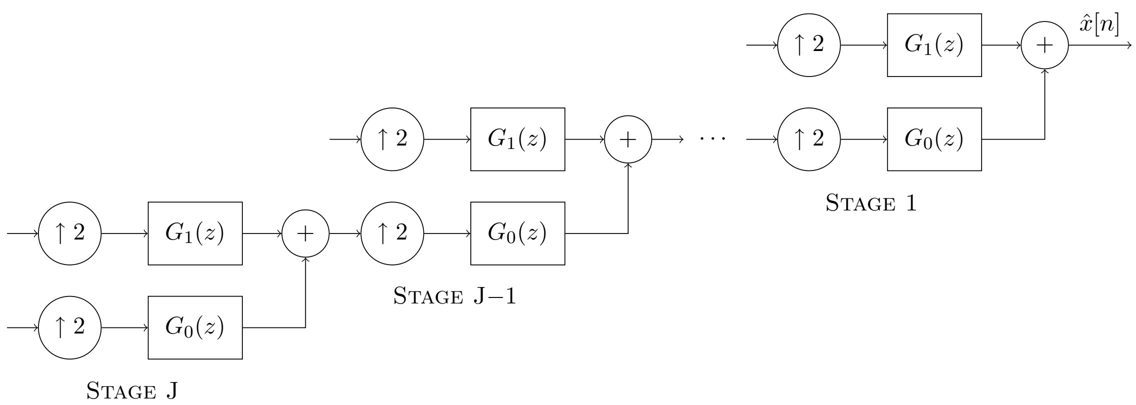

Figure 3: Reconstruction (synthesis) part of the tree structured filter bank whose analysis part is shown in Figure 2.

Let us consider what happens if the filters $h_i[n]$ and $g_i[n]$ in Figures 2 and 3 are the Haar filters, i.e., $g_i[n] = h_i[-n]$, $i = 0, 1$, and

$$G_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}), \qquad G_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1}).$$

Let us take for example $J = 3$, that is, we will split the original signal $x[n]$ three times. Before we continue, recall the multirate identity illustrated again in Figure 4. Using this

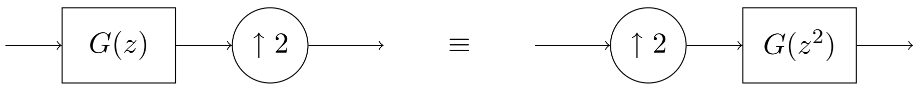

Figure 4: Upsampling identity revisited.

identity, we can write the equivalent filters of Figure 6 as represented in Figure 7 as follows.

$$\begin{aligned}
G_1^{(1)}(z) &= G_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1}) \\
G_1^{(2)}(z) &= G_0(z)G_1(z^2) = \frac{1}{2}(1 + z^{-1} - z^{-2} - z^{-3}) \\
G_1^{(3)}(z) &= G_0(z)G_0(z^2)G_1(z^4) = \frac{1}{2\sqrt{2}}(1 + z^{-1} + z^{-2} + z^{-3} - z^{-4} - z^{-5} - z^{-6} - z^{-7}) \\
&= \prod_{k=0}^{2} G_0(z^{2^k}).
\end{aligned}$$

In general,

$$\begin{aligned}
G_0^{(j)}(z) &= G_0^{(j-1)}(z)G_0(z^{2^{J-1}}) = \prod_{k=0}^{J-1} G_0(2^{2^k}) \\
G_1^{(j)}(z) &= G_0^{(j-1)}(z)G_1(z^{2^{j-1}}) = G_1(z^{2^{j-1}}) \prod_{k=0}^{j-2} G_0(z^{2^k}).
\end{aligned}$$

Now, let us examine how this works as a transform. Consider a two-channel orthogonal filter bank with analysis filters $h_0[n]$ and $h_1[n]$ and synthesis filters $g_i[n] = h_i[-n]$, $i = 0, 1$. Then we can write

$$\begin{aligned}
x[n] &= \sum_{k\in\mathbb{Z}} y_0^{(1)}[k]g_0^{(1)}[n - 2k] + \sum_{k\in\mathbb{Z}} y_1^{(1)}[k]g_1^{(1)}[n - 2k] \\
&= \sum_{k\in\mathbb{Z}} x_0^{(1)}[2k]g_0^{(1)}[n - 2k] + \sum_{k\in\mathbb{Z}} x_1^{(1)}[2k]g_1^{(1)}[n - 2k], \qquad (11)
\end{aligned}$$

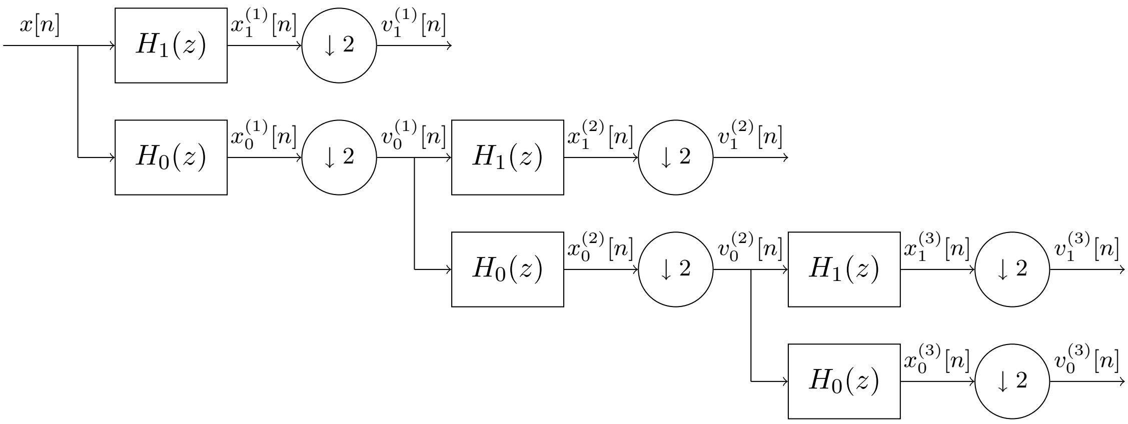

Figure 5: Analysis part of octave filter bank with $J = 3$ levels.

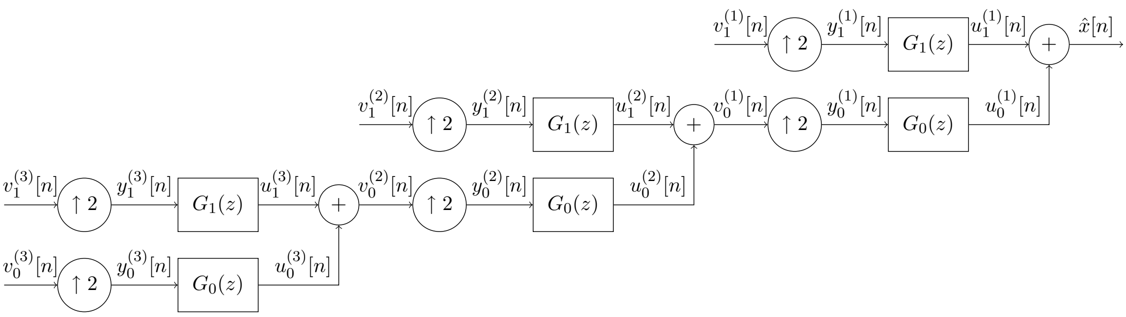

Figure 6: Synthesis part of octave filter bank with $J = 3$ levels.

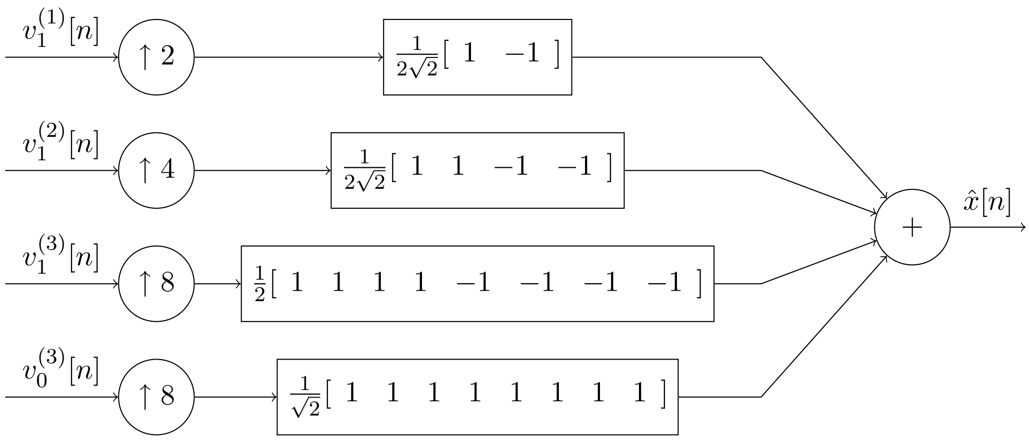

Figure 7: Octave-band synthesis filter bank, equivalent form for Haar-based two-channel filter bank.

where $g_i^{(1)}[n] = g_i[n]$, $i = 0, 1$. In the octave-band filter bank, only the low-pass branch is further split and therefore only the first term in (11) needs to be modified to account for the split in the filter bank. Therefore we can write the first term in (11) with $h_i^{(1)}[n] = g_i^{(1)}[-n]$, $i = 0, 1$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} x_0^{(1)}[2k]g_0^{(1)}[n-2k] &= \sum_{k\in\mathbb{Z}} x_0^{(1)}[2k]h_0^{(1)}[2k = n] \\ &= \sum_{k\in\mathbb{Z}} x_0^{(2)}[2^2k]g_0^{(2)}[n-2^2k] + \sum_{k\in\mathbb{Z}} x_1^{(2)}[2^2k]g_1^{(2)}[n-2^2k]\end{aligned}$$

where

$$\begin{aligned}G_0^{(2)}(z) &= G_0(z)G_0(z^2), \\ G_1^{(2)}(z) &= G_0(z)G_1(z^2), \\ H_i^{(2)}(z) &= G_i^{(2)}(z^{-1}), \quad i = 0, 1,\end{aligned}$$

and

$$\begin{aligned}x_0^{(2)}[2^2k] &= \langle h_0^{(2)}[2^2k-l], x[l]\rangle \\ x_1^{(2)}[2^2k] &= \langle h_1^{(2)}[2^2k-l], x[l]\rangle.\end{aligned}$$

If we proceed in this manner, we get

$$x[n] = \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} X^{j}[2k+1] g_1^{(j)}[n - 2^j k] + \sum_{k \in \mathbb{Z}} X^{(J)}[2k] g_0^{(J)}[n - 2^J k],$$

where

$$\begin{aligned} X^{(j)}[2k+1] &= \langle h_0^{(j)}[2^j k - l], x[l] \rangle, \quad j = 1, \cdots, J \\ X^{(J)}[2k] &= \langle h_0^{(J)}[2^J k - l], x[l] \rangle, \end{aligned}$$

and $h_i^{(j)}[n] = g_i^{(j)}[-n]$ as before. This is called the *discrete-time wavelet transform.*