Solutions: Homework Set # 5

Problem 1

(a) Since $h_3[n] = 0$, we have

$$\begin{aligned} h[n] &= h_2[n] * (\delta[n] + h_1[n]) \\ &= \alpha^n u[n] * (\delta[n] + \beta\delta[n-1]) \\ &= \alpha^n u[n] + \beta\alpha^{n-1} u[n-1]. \end{aligned}$$

(b) We can immediately write down

$$H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \beta \sum_{n=1}^{\infty} \alpha^{n-1} z^{-n}.$$

The first term clearly converges to $\frac{1}{1-\alpha z^{-1}}$ if $\left|\alpha z^{-1}\right| < 1$. The second term can be written as

$$\begin{aligned} \beta \sum_{n=1}^{\infty} \alpha^{n-1} z^{-n} &= \beta \sum_{k=0}^{\infty} \alpha^k z^{-k-1} \\ &= \beta z^{-1} \sum_{k=0}^{\infty} \alpha^k z^{-k} \\ &= \beta z^{-1} \frac{1}{1-\alpha z^{-1}}, \text{ if } \left|\alpha z^{-1}\right| < 1. \end{aligned}$$

Hence,

$$H(z) = \frac{1+\beta z^{-1}}{1-\alpha z^{-1}},$$

and the ROC is $|z| > |\alpha|$.

(c) We know that

$$\begin{aligned} Y(z) &= X(z)H(z) \\ \Rightarrow Y(z) &= X(z)\frac{1+\beta z^{-1}}{1-\alpha z^{-1}} \\ \Rightarrow Y(z)(1-\alpha z^{-1}) &= X(z)(1+\beta z^{-1}) \\ \Rightarrow y[n] - \alpha y[n-1] &= x[n] + \beta x[n-1], \end{aligned}$$

where the last line follows from the fact that $z^{-1}X(z)$ is the Z-transform of $x[n-1]$ for any signal $x[n]$.

(d) Yes, the system is causal, because $h[n] = 0$ for all $n < 0$.

(e) Note that the ROC of the system function is $|z| > |\alpha|$. Since $|\alpha| < 1$, we see that the ROC contains the unit circle, and therefore, the DTFT is summable and the system is stable.

(f) Here, the functions $h_1[n]$, $h_2[n]$ and $h_3[n]$ are not the same as in parts (a) through (e). Since $h_3[n]$ is non-zero, it is difficult to write down the transfer function $h[n]$ directly. It is easier to derive the difference equation as follows.

$$\begin{aligned} y[n] &= (x[n] + 2x[n-1] + \frac{1}{3}y[n-1]) * h_2[n] \\ &= \frac{1}{2}x[n-1] + x[n-2] + \frac{1}{6}y[n-2]. \end{aligned}$$

Thus, the difference equation is

$$y[n] - \frac{1}{6}y[n-2] = \frac{1}{2}x[n-1] + x[n-2].$$

From this, we can easily find the Z-transform of $h[n]$:

$$H(z) = \frac{\frac{1}{2}z^{-1} + z^{-2}}{1 - \frac{1}{6}z^{-2}}.$$

Now all we need to do is take the inverse Z-transform of $H(z)$ to find $h[n]$. To do this, we first note that the degree of the numerator and the denominator are equal. Hence, we can apply a long division to reduce the degree of the numerator:

$$(z^{-2} + \frac{1}{2}z^{-1}) : (-\frac{1}{6}z^{-2} + 1) = -6 + \frac{-6 + \frac{1}{2}z^{-1}}{1 - \frac{1}{6}z^{-2}}.$$

To be able to take the inverse Z-transform, we need to simplify the degree-2 term in the denominator. We use partial fraction expansion to do so. The poles of $H(z)$ are the zeros of $1 - \frac{1}{6}z^{-2}$, which are $p_1 = \frac{1}{\sqrt{6}}$ and $p_2 = -\frac{1}{\sqrt{6}}$. Hence,

$$1 - \frac{1}{6}z^{-2} = (1 - p_1 z^{-1})(1 - p_2 z^{-1}) = (1 - \frac{1}{\sqrt{6}}z^{-1})(1 + \frac{1}{\sqrt{6}}z^{-1}).$$

We then find A and B such that

$$\begin{aligned} \frac{-6 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{\sqrt{6}}z^{-1})(1 + \frac{1}{\sqrt{6}}z^{-1})} &= \frac{A}{1 - \frac{1}{\sqrt{6}}z^{-1}} + \frac{B}{1 + \frac{1}{\sqrt{6}}z^{-1}} \\ \Rightarrow -6 + \frac{1}{2}z^{-1} &= A + A\frac{1}{\sqrt{6}}z^{-1} + B - B\frac{1}{\sqrt{6}}z^{-1}. \end{aligned}$$

Comparing the constant terms and the coefficients of z^{-1} separately, we obtain the two equations

$$\begin{aligned} \text{I: } A + B &= -6 \\ \text{II: } A - B &= \frac{\sqrt{6}}{2}. \end{aligned}$$

Solving this yields $A = \frac{\sqrt{6}}{4} - 3$, $B = -\frac{\sqrt{6}}{4} - 3$. Thus, we can write

$$H(z) = \frac{\frac{\sqrt{6}}{4} - 3}{1 - \frac{1}{\sqrt{6}}z^{-1}} + \frac{-\frac{\sqrt{6}}{4} - 3}{1 + \frac{1}{\sqrt{6}}z^{-1}} - 6.$$

Now, since both poles of $H(z)$ have the same magnitude, we have only two choices of the ROC: $|z| < \frac{1}{\sqrt{6}}$ or $|z| > \frac{1}{\sqrt{6}}$. If we look at the system diagram, we can see that an incoming signal is only delayed by $\delta[n-1]$ functions, and hence, the system must be causal. If we want the resulting $h[n]$ to be causal, we should choose the ROC $|z| > \frac{1}{\sqrt{6}}$, and we obtain

$$h[n] = (\frac{\sqrt{6}}{4} - 3)\left(\frac{1}{\sqrt{6}}\right)^n u[n] + (-\frac{\sqrt{6}}{4} - 3)\left(\frac{1}{\sqrt{6}}\right)^n u[n] - 6\delta[n].$$

Note that since $\frac{1}{\sqrt{6}} < 1$, we can even conclude that the system considered in part (f) is stable.

Problem 2

(a) We have $h[n] = nw[n] + w[n]$, and so, using the standard properties of the Z-transform,

$$H(z) = -z\frac{dW(z)}{dz} + W(z).$$

The ROC of $H(z)$ is the same as that of $W(z)$: $\mathcal{R}_H = \mathcal{R}_W$.

(b) The Z-transform of $w[n]$ is

$$\begin{aligned} W(z) &= \sum_{n=1}^{\infty}(az^{-1})^n \\ &= \frac{1}{1-az^{-1}} - 1 \\ &= \frac{az^{-1}}{1-az^{-1}}. \end{aligned}$$

$W(z)$ has a single pole in $z = a$; since $w[n]$ is causal, the ROC extends outwards, so $\mathcal{R}_W = \{z : |z| > a\}$.

(c) The DTFT exists if the ROC contains the unit circle. This is the case here if $a < 1$.

(d) From (a), $h[n] = na^n u[n-1] + a^n u[n-1]$.

(e) $G(z)$ can be seen as the concatenation of two systems $G_1(z)$ and $G_2(z)$, where $G_1(z) = 1/H(z)$ is the inversion of $H(z)$ and $G_2(z) = z^{-2}$ adds a two-tap delay. Thus, $G(z) = z^{-2}/H(z)$.

(f) Let us compute $H(z)$. We have

$$\begin{aligned} H(z) &= W(z) - z\frac{dW(z)}{dz} \\ &= \frac{az^{-1}}{1-az^{-1}} - z\frac{d}{dz}\frac{az^{-1}}{1-az^{-1}} \\ &= \frac{az^{-1}}{1-az^{-1}} + \frac{az^{-1}}{(1-az^{-1})^2} \\ &= \frac{2az^{-1}(1-\frac{1}{2}az^{-1})}{(1-az^{-1})^2}. \end{aligned}$$

Thus,

$$G(z) = \frac{z^{-2}}{H(z)} = \frac{z^{-1}(1-az^{-1})^2}{2a(1-\frac{1}{2}az^{-1})}.$$

We see that $G(z)$ has a single pole at $z = a/2$. In theory, we have two choices for the ROC of $G(z)$, leading to two different inverse Z-transforms (two different systems). The anticausal version would be to choose the ROC $|z| < \frac{a}{2}$. However, in that case, there is a problem. The combined system given in the Figure is a working system, with a transfer function of $\delta[n]$. The system function in the Z domain is $H(z)G(z)$, with a ROC which is the intersection of the ROC of $H(z)$ and the ROC of $G(z)$. But if we take the ROC of $G(z)$ to be $|z| < \frac{a}{2}$, then the intersection is empty (draw a picture to see this).

Therefore, we have to choose the other option for the ROC of $G(z)$, namely $|z| > \frac{a}{2}$. In that case, the intersection is non-empty, and the overall ROC of $H(z)G(z)$ is $|z| > a$. Hence, the overall system is causal, which makes sense. The system corresponding to $G(z)$ alone has to be causal, too.

Now, we see that $G(z)$ can be stable if and only if $a < 2$.

Let us determine $g[n]$ (you didn't have to do this for the homework). Using long division, we find that

$$G(z) = -z^{-2} - a^{-2} + \frac{a^{-2}}{1-\frac{1}{2}az^{-1}}.$$

By inspection of the terms, we see that the inverse Z-transform is

$$g[n] = -\delta[n-2] - a^{-2}\delta[n] + a^{-2}(a/2)^n u[n].$$

Problem 3

(a) A system is stable if the unit-circle lies inside the region of convergence (ROC). If we are given a pole-zero plot of a system function, we have at least two choices for the ROC. If we choose the ROC to be all complex numbers z such that $|z| > |p_i|$, where p_i is the pole with the largest absolute value, then the corresponding system is causal. If we choose the ROC to be all z such that $|z| < |p_k|$, where p_k is the pole with the smallest absolute value, then the corresponding system is anticausal. If, however, we choose the ROC to be all z such that $|p_j| < |z| < |p_l|$, then the corresponding system is two-sided. We can see that we have the following correspondences:

- (i): 2.
- (ii): 1.
- (iii): 1.
- (iv): 3.

(b) Since the inverse system function is $G(z) = \frac{1}{H(z)}$, we see that

$$G(z) = \frac{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})}{(1-z_1z^{-1})(1-z_2z^{-1})},$$

and it becomes clear that the poles of $H(z)$ are the zeros of $G(z)$ and vice versa. Hence, we argue as in part (a), but using the zeros instead of the poles. Hence, we are in case 3.

Problem 4

(a)

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} = \frac{z(3z-4)}{z^2 - 3.5z^1 + 1.5}$$

Zeros are roots of nominator and poles are roots of denominator polynomials. Hence, the zeros are $z_1 = 0$ and $z_2 = \frac{4}{3}$ and the poles are $p_1 = 0.5$ and $p_2 = 3$.

(b)

$$\begin{aligned} H(z) &= \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} \\ &= \frac{3 - 4z^{-1}}{(1 - 0.5z^{-1})(1 - 3z^{-1})} \\ &= \frac{1}{1 - 0.5z^{-1}} + \frac{2}{(1 - 3z^{-1})} \end{aligned}$$

(c) Since ROC does not contain any poles, there exists three possible ROC for $H(z)$, which are the following: $0 \le z < 0.5$, $0.5 < z < 3$ and $z > 3$. In a stable system, ROC contains the unit circle, therefore ROC is $0.5 < z < 3$. In an anticausal system, ROC contains zero, therefore ROC is $0 \le z < 0.5$.

(d) Impulse response of any stable system is absolutely summable and absolutely summability is a necessary condition in existence of DTFT. In this case, for the sequence with ROC $0.5 < z < 3$, DTFT is computable.

Problem 5

(a)

$$y[n] = 0.5y[n-1] + x[n] - 3x[n-1]$$

The system is causal (by zero initial condition assumption) and by Taking Z-transform for the above input-output relation, we have

$$Y(z) = 0.5z^{-1}Y(Z) + X(Z) - 3z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 3z^{-1}}{1 - 0.5z^{-1}}$$

.

(b) $H(z)$ has a pole at $p = 0.5$ and a zero at $z = 3$. The zero is outside of unit circle.

$$\begin{aligned} H(z) &= 3\frac{\frac{1}{3} - z^{-1}}{1 - 0.5z^{-1}} \\ &= 3\frac{\frac{1}{3}z^{-1} - 1}{1 - 0.5z^{-1}} \frac{\frac{1}{3} - z^{-1}}{\frac{1}{3}z^{-1} - 1} \end{aligned}$$

$$H_{min}(z) = \frac{z^{-1} - 3}{1 - 0.5z^{-1}}$$

$$H_{ap}(z) = \frac{\frac{1}{3} - z^{-1}}{\frac{1}{3}z^{-1} - 1}$$

(c) Since the system is causal the ROC is $Z > 0.5$ and because ROC contains the unit circle, the system is BIBO stable.

(d)

$$Y(z) = H(z)X(z)$$
$$X(z) = \frac{1}{H(z)}Y(z)$$
$$R(z) = \frac{1}{H(z)}$$

$R(z)$ has a zero at $z = 0.5$ and a pole at $p = 3$. The inverse system cannot be both causal and stable. Because if the system is causal the ROC is $z > 3$ which imply that system is not stable and if system is stable and the ROC contains the unit circle $z < 3$, the system is anticausal.

(d)

$$G(z) = H(z)\frac{1}{H_{min}(z)} = H_{min}(z)H_{ap}(z)\frac{1}{H_{min}(z)} = H_{ap}(z)$$

The amplitude of an all-pass filter is one.

$$|G_{ap}(z)|_{z=e^{j\omega}} = 1$$

$$\begin{aligned}
\arg(G(z))_{z=e^{j\omega}} &= \arg(H_{ap}(z))_{z=e^{j\omega}} \\
&= \arg\left(\frac{\frac{1}{3} - e^{-j\omega}}{\frac{1}{3}e^{-j\omega} - 1}\right) \\
&= \arg\left(e^{-j\omega}\frac{\frac{1}{3}e^{j\omega} - 1}{\frac{1}{3}e^{-j\omega} - 1}\right) \\
&= -\omega + \arg\left(\frac{\frac{1}{3}e^{j\omega} - 1}{\frac{1}{3}e^{-j\omega} - 1}\right) \\
&= -\omega + 2\arg\left(\frac{1}{3}e^{j\omega} - 1\right) \\
&= -\omega + 2\tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega) - 3}\right)
\end{aligned}$$