EPFL Winter Semester 2007/2008
Handout # 14, Tuesday, November 6, 2007

Solutions: Homework Set # 4

Problem 1

(a) Let $a[n] = x[-n]$

$$\begin{aligned} A(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n} \\ &= X(e^{-j\omega}) \end{aligned}$$

(b) As $x[n]$ is real $x^*[n] = x[n]$:

$$\begin{aligned} X^*(e^{-j\omega}) &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* \\ &= \sum_{n=-\infty}^{\infty} x^*[n]\left(e^{j\omega n}\right)^* \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= X(e^{j\omega}) \end{aligned}$$

(c) If a complex number $z = x + jy$ is such that $z = z^*$ then z is real:

$$\begin{aligned} z &= x + jy \\ &= z^* \\ &= x - jy \\ &\Longrightarrow x + jy = x - jy \\ &\Longrightarrow y = 0 \end{aligned}$$

We use the properties that $x[n] = x[-n]$ and that $x[n]$ is real and we show that $X^*(e^{j\omega})$

$= X(e^{j\omega})$:

$$\begin{aligned}
X^*(e^{j\omega}) &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right)^* \\
&= \sum_{n=-\infty}^{\infty} x^*[n] \left(e^{-j\omega n}\right)^* \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x[-n]e^{j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= X(e^{j\omega})
\end{aligned}$$

(d) If a complex number $z = x + jy$ is such that $-z = z^*$ then z is imaginary:

$$\begin{aligned}
-z &= -x - jy \\
&= z^* \\
&= x - jy \\
&\Longrightarrow -x - jy = x - jy \\
&\Longrightarrow x = 0
\end{aligned}$$

We use the properties that $x[n] = -x[-n]$ and that $x[n]$ is real and we show that $X^*(e^{j\omega})$ $= -X(e^{j\omega})$:

$$\begin{aligned}
X^*(e^{j\omega}) &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right)^* \\
&= \sum_{n=-\infty}^{\infty} x^*[n] \left(e^{-j\omega n}\right)^* \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \\
&= -\sum_{n=-\infty}^{\infty} x[-n]e^{j\omega n} \\
&= -\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= -X(e^{j\omega})
\end{aligned}$$

Problem 2

(a) We recall from homework 1 that $\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$.Now,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} n2^{-n}u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} n2^{-n}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} n\left(\frac{e^{-j\omega}}{2}\right)^n \\ &= \frac{\frac{e^{-j\omega}}{2}}{\left(1-\frac{e^{-j\omega}}{2}\right)^2} \\ &= \frac{e^{-j\omega}}{2\left(1-\frac{e^{-j\omega}}{2}\right)^2} \end{aligned}$$

(b) We first calculate the DTFT of $2^{-n}u[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} \left(\frac{e^{-i\omega n}}{2}\right)^n \\ &= \frac{1}{1-\frac{e^{-j\omega}}{2}} \end{aligned}$$

Now,

$$j\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{1-\frac{e^{-j\omega}}{2}} = \frac{e^{-j\omega}}{2\left(1-\frac{e^{-j\omega}}{2}\right)^2}$$

We can see that answers parts a) and b) are the same.

Problem 3

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}\right)e^{-j\omega n} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1} e^{j(\frac{2\pi k}{N}-\omega)n} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\frac{1-e^{-j\omega N}}{1-e^{j(\frac{2\pi k}{N}-\omega)}} \\
&= \frac{1-e^{-j\omega n}}{N}\sum_{k=0}^{N-1}\frac{X[k]}{1-e^{j(\frac{2\pi k}{N}-\omega)}}
\end{aligned}$$

Problem 4

(a) False. Consider for example:

$$y[n] = \frac{1}{n}x[n]$$

(b) True:

$$\begin{aligned}
y[n+N] &= H(x[n+N]) \quad \text{(time-invariance)} \\
&= H(x[n]) \quad \text{(periodicity of } x[n]\text{)} \\
&= y[n]
\end{aligned}$$

(c) 1.

$$y[n] = g[n]x[n]$$

Stability: Suppose $|x[n]| < A \ \forall n$.

$$\begin{aligned}
|y[n]| &= |g[n]||x[n]| \\
&< A|g[n]|
\end{aligned}$$

This is stable only if $g[n]$ is bounded.
Causality: Yes, indeed, $y[n]$ depends only on the current input.
Linearity:

$$\begin{aligned}
g[n](ax_1[n]+bx_2[n]) &= ag[n]x_1[n]+bg[n]x_2[n] \\
&= ay_1[n]+by_2[n] \\
&\Longrightarrow \text{linear}
\end{aligned}$$

Time-invariance: for a general $g[n]$ the system is not time-invariant

$$\begin{aligned} g[n]x[n-n_0] &\neq g[n-n_0]x[n-n_0] \\ &= y[n-n_0] \end{aligned}$$

2.

$$y[n] = \sum_{k=n_0}^{n} x[k]$$

Stability: Suppose $|x[k]| = c$ a constant.

$$\begin{aligned} |y[n]| &= |\sum_{k=n_0}^{n} x[k]| \\ &\leq \sum_{k=n_0}^{n} |x[k]| \\ &= c(n-n_0+1) \end{aligned}$$

We can see that there does not exist B such that $(n-n_0+1)c < B \; \forall n$, therefore the system is not stable.
Causality: Yes, indeed, $y[n]$ does not depend on the future.
Linearity:

$$\begin{aligned} \sum_{k=n_0}^{n} (ax_1[n] + bx_2[n]) &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= ay_1[n] + by_2[n] \\ &\Longrightarrow \text{linear} \end{aligned}$$

Time-invariance: The system is not time-invariant

$$\begin{aligned} \sum_{k=n_0}^{n} x[k-k_0] &= \sum_{k=n_0-k_0}^{n-k_0} x[k] \\ &\neq \sum_{k=n_0}^{n-k_0} x[k] \\ &= y[n-n_0] \end{aligned}$$

3.

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

Stability: Suppose $|x[k]| < A \ \forall n$. Then,

$$\begin{aligned} |y[n]| &= \left|\sum_{k=n-n_0}^{n+n_0} x[k]\right| \\ &\leq \sum_{k=n-n_0}^{n+n_0} |x[k]| \\ &< (2n_0+1)A \\ &\Longrightarrow \text{stable} \end{aligned}$$

Now let $h[n] = 1$ for $-n_0 \leq n \leq n_0$ and 0 otherwise.
One can verify that $y[n] = (h * x)[n]$.
It follows that $y[n]$ is not causal as there are value(s) of $n < 0$ such that $h[n] \neq 0$.
It also follows that $y[n]$ is linear and time invariant.

4.

$$y[n] = ax[n] + b$$

Stability: Suppose $|x[k]| < A \ \forall n$. Then,

$$\begin{aligned} |y[n]| &= |ax[n]+b| \\ &\leq |a||x[n]| + |b| \\ &< |a|A + |b| \\ &\Longrightarrow \mathit{stable} \end{aligned}$$

Causality: Yes, indeed, $y[n]$ does not depend on the future.
Linearity: The system is not linear

$$\begin{aligned} a(\alpha x_1[n] + \beta x_2[n]) + b &= a\alpha x_1[n] + a\beta x_2[n] + b \\ &\neq \alpha(ax_1[n]+b) + \beta(ax_2[n]+b) \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Time-invariance: The system time-invariant

$$x[n-n_0] + b = y[n-n_0]$$

Problem 5

Take any impulse response $h[n]$ of an LTI system. Then,

$$\begin{aligned} y[n] &= (h * x)[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]a^{n-k} \\ &= \sum_{k=-\infty}^{\infty} h[k]a^{n}\frac{1}{a^k} \\ &= a^n\left(\sum_{k=-\infty}^{\infty} h[k]a^{-k}\right) \\ &= a^n K \end{aligned}$$

Where K is a constant which does not depend on n.
Therefore the eigenvalue is:

$$\textstyle\sum_{k=-\infty}^{\infty} h[k]a^{-k}$$

Problem 6

(a) Observing the figure we deduce that: $h[n] = (h_5 * h_1)[n] + (h_2 * h_1)[n] - (h_4 * h_3 * h_1)[n]$.

(b) We first calculate $(h_5 * h_1)[n]$:

$$\begin{aligned} (h_5 * h_1)[n] &= \sum_{k=-\infty}^{\infty} h_1[k]h_5[n-k] \\ &= nu[n] - \frac{1}{2}(n-1)u[n-1] \end{aligned}$$

We then we calculate $(h_2 * h_1)[n]$:

$$\begin{aligned} (h_2 * h_1)[n] &= \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k] \\ &= 2nu[n-1] - (n-1)u[n-2] \end{aligned}$$

We lastly calculate $h_4 * h_3 * h_1[n]$:

$$\begin{aligned} (h_4 * h_3 * h_1)[n] &= h_4 * (\sum_{k=-\infty}^{\infty} h_1[k]h_3[n-k]) \\ &= h_4 * (n-2)u[n-1] - \frac{1}{2}(n-3)u[n-2]) \\ &= (n-3)u[n-2] - \frac{1}{2}(n-4)u[n-3]) \end{aligned}$$

We now sum up all the intermediate results to get:

$$h[n] = nu[n] + \frac{1}{2}(3n-1)u[n-1] - (2n-4)u[n-2] + \frac{1}{2}(n-4)u[n-4]$$

(c) We get:

$$(h * x)[n] = 2h[n] + h[n-2] - 3h[n-3],$$

where $h[n]$ is the result from part (b).