Solutions: Homework Set # 3

Problem 1 (DTFT)

(a) Compute the DTFT of $x[n] = u[n-2] - u[n-4]$.

$$x[n] = \delta[n-2] + \delta[n-3]$$
$$X(e^{j\omega}) = e^{-j2w} + e^{-j3w}$$

(b) Compute the DTFT of $h[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$.

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{jn\omega} \\ &= \frac{0.5e^{-j\omega}}{1-0.5e^{-j\omega}} \end{aligned}$$

(c) Compute the DTFT of $y[n] = x[n] * h[n] = \sum_k x[k]h[n-k]$.

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]e^{jn\omega} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{jn\omega} \\ &= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[k]h[n-k]e^{j(n-k+k)\omega} \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{jk\omega} \sum_{n=-\infty}^{\infty} h[n-k]e^{j(n-k)\omega} \\ &= X(e^{j\omega})H(e^{j\omega}) \end{aligned}$$

(d) Compute the DTFT of $z[n] = y[n-n_0]$.

$$\begin{aligned} Z(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} z[n]e^{jn\omega} \\ &= \sum_{n=-\infty}^{\infty} y[n-n_0]e^{jn\omega} \\ &= \sum_{n=-\infty}^{\infty} y[n-n_0]e^{j(n-n_0+n_0)\omega} \\ &= e^{jn_0\omega} \sum_{n=-\infty}^{\infty} y[n-n_0]e^{j(n-n_0)\omega} \\ &= e^{jn_0\omega} Y(e^{j\omega}) \end{aligned}$$

Problem 2 (DFS)

(a) The periods of both sequences are 6. The computation is as follows

$$N = k\frac{2\pi}{\omega} = 6k = 6$$

By definition, we have:

$$\tilde{x}[n] = 1 + \cos\left(\frac{2\pi}{6}n\right) = 1 + 0.5e^{j\frac{2\pi}{6}n} + 0.5e^{-j\frac{2\pi}{6}n}$$

$$\tilde{X}[k] = \begin{cases} 6 & k = 0 \\ 3 & k = 1,5 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{y}[n] = \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) = \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}$$

$$\tilde{Y}[k] = \begin{cases} -3je^{j\frac{\pi}{4}} & k = 1 \\ 3je^{-j\frac{\pi}{4}} & k = 5 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned}
\tilde{W}[k] &= \sum_{n=0}^{N-1} \tilde{w}[n]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \tilde{u}[n]\tilde{v}[n]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} \tilde{v}[n]\tilde{U}[l]e^{j\frac{2\pi}{N}ln}e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{U}[l]\sum_{n=0}^{N-1} \tilde{v}[n]e^{-j\frac{2\pi}{N}(k-l)n} \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{U}[l]\tilde{V}[k-l]
\end{aligned}$$

(c)

$$\begin{aligned}
\tilde{R}[k] &= \sum_{n=0}^{N-1} \tilde{r}[n]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \tilde{W}[n]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} \tilde{U}[l]\tilde{V}[n-l]e^{-j\frac{2\pi}{N}k(n-l+l)} \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{U}[l]e^{-j\frac{2\pi}{N}kl}\sum_{n=0}^{N-1} \tilde{V}[n-l]e^{-j\frac{2\pi}{N}k(n-l)} \\
&= N\tilde{v}[-k]\tilde{u}[-k]
\end{aligned}$$

(d) In this part we can compute the DFS either using the expression derived in part (b) or directly. Note that

$$2\sin(a)\cos(b) = sin(a+b) + sin(a-b)$$

$$\begin{aligned}
\tilde{z}[n] &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)\cos\left(\frac{2\pi}{6}n\right) \\
\tilde{z}[n] &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + 0.5\sin\left(\frac{4\pi}{6}n + \frac{\pi}{4}\right) + 0.5\sin\left(\frac{\pi}{4}\right) \\
&= \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n} + 0.5\sin\left(\frac{\pi}{4}\right) + \frac{1}{4j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}2n} - \frac{1}{4j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}2n}
\end{aligned}$$

$$\tilde{Z}[k] = \begin{cases} 3\sin\left(\frac{\pi}{4}\right) & k=0 \\ -3je^{j\frac{\pi}{4}} & k=1 \\ -1.5je^{j\frac{\pi}{4}} & k=2 \\ 0 & k=3 \\ 1.5je^{-j\frac{\pi}{4}} & k=4 \\ 3je^{-j\frac{\pi}{4}} & k=5 \end{cases}$$

Problem 3

(a)

1. This statement is wrong. It is not possible to have $X[k] = 0$ for all $k = 0, \ldots, M-1$ if $M \geq N$. We proof that this cannot be true. The proof works by contradiction. Assume that $X[k] = 0$ for all $k = 0, \ldots, M-1$ and that $M \geq N$. Also, assume that $x[n]$ is as described in this problem.
   Then, because $M \geq N$, we can write

   $$\begin{aligned} X[k] &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi}{M}kn} \\ &= \sum_{n=0}^{M-1} x[n]e^{-j\frac{2\pi}{M}kn}. \end{aligned}$$

   This last expression is the discrete Fourier transform (DFT) of the M-length sequence $\big(x[0], x[1], \ldots, x[N-1], 0, 0, \ldots, 0\big)$, where the zeros are added to make the sequence-length to be equal to M. Now, since we assume that $X[k] = 0$ for $k = 0, \ldots, M-1$, and since the DFT is a one-to-one mapping, we conclude that

   $$\big(x[0], x[1], \ldots, x[N-1], 0, 0, \ldots, 0\big) = \big(0, 0, \ldots, 0\big).$$

   From this, it follows that $x[n] = 0$ for $n = 0, \ldots, N-1$. But this is a contradiction with the assumption that $x[n]$ is *non-zero*. Hence, the statement is wrong.

2. This statement is true. We can construct a sequence $x[n]$ that satisfies the claim as follows. Since we have $M < N$, we will, for simplicity, choose $M = 1$. We see that $X[k]$ is a sampled version of the discrete-time Fourier transform (DTFT) of $x[n]$, where the samples are uniformly spaced in $[0, 2\pi]$. If $M = 1$, we only take one sample at $\omega = \frac{2\pi}{M}0 = 0$. Hence, all we need is a sequence $x[n]$ of finite length N larger than $M = 1$, whose DTFT is such that $X(e^{j\omega}) = 0$ for $\omega = 0$. For example, we can choose $X(e^{j\omega}) = \sin(\omega)$. The sequence $x[n]$ that corresponds to this sequence is

   $$x[n] = \begin{cases} \frac{1}{2j} & \text{if } n = -1, \\ -\frac{1}{2j} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This sequence has support $N = 3$, but it is non-zero for a negative value of n. So, we have to slightly modify it by shifting it to the right. The new choice is

$$x[n] = \begin{cases} \frac{1}{2j} & \text{if } n = 0, \\ -\frac{1}{2j} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding DTFT is

$$X(e^{j\omega}) = e^{-j\omega}\sin(\omega).$$

If we sample the above $X(e^{j\omega})$ at $\omega = 0$, we obtain 0. Hence, if $M = 1$, the sequence $X[k]$ (of length 1) is all-zero. However, the sequence $x[n]$ is not all-zero.

(b) $X_1[k]$ is simply the DFT of $x[n]$. Let us now analyze $X_2[k]$, whereby we study the two cases where k is even and k is odd separately. First, we have

$$\begin{aligned} X_2[2k] &= \sum_{n=0}^{M-1} x[n]e^{-j\frac{2\pi}{2N}2kn} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\ &= X_1[k]. \end{aligned}$$

Now,

$$\begin{aligned} X_2[2k+1] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}(k+\frac{1}{2})n} \\ &= \text{“}X_1[k+\tfrac{1}{2}]\text{”}, \end{aligned}$$

where we put the last expression in quotes since $X_1[k]$ is not actually defined for $k \notin \mathbb{Z}$. As you should recall, such a noninteger shift can be obtained by taking the DFT of the sequence $W_N^{n/2}x[n]$. Another way to interpret this is to remember that the DFT coefficients correspond to samples of the (continuous) DTFT, evaluated at $\omega = (2\pi/N)k$, $k \in \mathbb{Z}$. Noninteger shifts of the DFT then can be obtained by sampling the DTFT for noninteger k.

Graphically, $X_2[k]$ looks like a “smoothed”, or interpolated version of $X_1[k]$ (cf. also the solutions to Homework 2, Problem 5(b)).

Problem 4

(a) It is easy to see that $\langle \tilde{h}_i, \tilde{h}_j \rangle = 0$, for $i \neq j$, i.e., $\tilde{H}$ is orthogonal. For example

$$\begin{aligned} \langle \tilde{h}_2, \tilde{h}_7 \rangle &= \sum_{k=1}^{8} \tilde{h}_2(k)\tilde{h}_7^*(k) \\ &= 1\cdot 1 + (-1)\cdot 1 + 1\cdot(-1) + (-1)\cdot(-1) + 1\cdot(-1) + (-1)\cdot(-1) + 1\cdot 1 + (-1)\cdot 1 = 0. \end{aligned}$$

However, the norm of the vectors is not 1:

$$\| \tilde{\{h\}}_i \| = \langle \tilde{h}_i, \tilde{h}_i \rangle^{1/2} = \left(\sum_{k=1}^{8} 1 \right)^{1/2} = \sqrt{8}.$$

Therefore, in order to make the basis orthonormal, we need to normalize the vectors by their norm. This gives us $H = \{h_1, \ldots, h_8\}$, where $h_i = \frac{1}{\sqrt{8}}\tilde{h}_i$ for $i = 1, \ldots, 8$.

(b)

$$\begin{aligned}\mathbf{a}_H &= [2, 0, -1, 4, -3, 1, 1, 0] \\ &\Downarrow \\ \mathbf{a}_E &= 2h_1 + 0h_2 + (-1)h_3 + 4h_4 + (-3)h_5 + 1h_6 + 1h_7 + 0h_8 \\ &= \left[\begin{array}{cccccccc} 1.4142 & -2.1213 & -1.4142 & 0.7071 & 2.1213 & 0 & 0.7071 & 4.2426 \end{array}\right]\end{aligned}$$

Note that this also can be computed using the matrix multiplication

$$\begin{aligned}\mathbf{a}_H \cdot \mathbf{H} &= \left[\begin{array}{cccccccc} 2 & 0 & -1 & 4 & -3 & 1 & 1 & 0 \end{array}\right] \cdot \frac{1}{\sqrt{8}} \left[\begin{array}{rrrrrrrr} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]_E \\ &= \left[\begin{array}{cccccccc} 1.4142 & -2.1213 & -1.4142 & 0.7071 & 2.1213 & 0 & 0.7071 & 4.2426 \end{array}\right]\end{aligned}$$

(c) Using the same transform as in (b), we can write the measured vector b_H in E basis.

$\mathbf{b}_E = \mathbf{b}_H \cdot \mathbf{H} = \left[\begin{array}{cccccccc} 2.8284 & -8.4853 & 2.8284 & 2.8284 & -8.4853 & 5.6569 & -2.8284 & 5.6569 \end{array}\right].$

Since the expected measured vectors has zeros in the last two elements, the best estimation is obtained by replacing the corresponding elements by zeros:

$$\hat{\mathbf{b}}_E = \left[\begin{array}{cccccccc} 2.8284 & -8.4853 & 2.8284 & 2.8284 & -8.4853 & 5.6569 & 0 & 0 \end{array}\right],$$

which is

$$\hat{\mathbf{b}}_H = \hat{\mathbf{b}}_E \mathbf{H}^{-1} = \left[\begin{array}{cccccccc} -1 & -1 & -5 & -1 & 1 & 9 & -3 & 9 \end{array}\right]$$

in H-basis. Note that $\mathbf{H} = \mathbf{H}^{-1}$, since H forms an orthonormal basis. You can easily check that

$$|\mathbf{b}_E - \hat{\mathbf{b}}_E| = |\mathbf{b}_H - \hat{\mathbf{b}}_H| = \sqrt{40}.$$

(d) According to the definition of $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ we have

$$\begin{aligned}\hat{\mathbf{x}}_1 &= \left[\begin{array}{cccccccc} 5 & 2 & -7 & 8 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow |\mathbf{x} - \hat{\mathbf{x}}_1|^2 = 10^2 + 3^2 + (-2)^2 = 113 \\ \hat{\mathbf{x}}_2 &= \left[\begin{array}{cccccccc} 5 & 0 & -7 & 8 & 0 & 10 & 3 & 0 \end{array}\right] \quad \Rightarrow |\mathbf{x} - \hat{\mathbf{x}}_2|^2 = 2^2 + 0^2 + (-2)^2 = 8 \\ \hat{\mathbf{x}}_3 &= \left[\begin{array}{cccccccc} 5 & 2 & 0 & 8 & 0 & 10 & 3 & 0 \end{array}\right] \quad \Rightarrow |\mathbf{x} - \hat{\mathbf{x}}_1|^2 = (-7)^2 + 0^2 + (-2)^2 = 53.\end{aligned}$$

Clearly, $\hat{x}_2$ has the smallest reconstruction error.

(e) Assume we want to store only k elements out of n elements of the vector $\mathbf{x} = \left[\begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array}\right]$. Let $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ be the element which are stored, and $x_{i_{k+1}}, \dots, x_{i_n}$ be the missing elements. Clearly,

$$|\mathbf{x} - \hat{\mathbf{x}}|^2 = \sum_{j=k+1}^{n} x_{i_j}^2.$$

In order to minimize this error, one has to choose the k element with the largest absolute value to be stored, and throw away the remaining $n-k$ elements with the smallest absolute value:

$$|x_{i_1}| \geq |x_{i_2}| \geq \cdots \geq |x_{i_k}| \geq |x_{i_{k+1}}| \geq \cdots \geq |x_{i_1}|.$$