EPFL Winter Semester 2007/2008
Handout # 11, Monday, 22 October, 2007

# Solutions: Homework Set # 2

## Problem 1

We are looking for $N \in \mathbb{N}$ such that for all $n$, $\tilde{x}[n+N] = \tilde{x}[n]$. This means that we need to find $N$ for which

$$2 + \sin\left(\frac{2\pi}{5}(n+N)\right) + \cos\left(\frac{3\pi}{2}(n+N)\right) = 2 + \sin\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{3\pi}{2}n\right). \tag{1}$$

We have that $\sin\left(\frac{2\pi}{5}(n+N_1)\right) = \sin\left(\frac{2\pi}{5}n\right)$ for $N_1 = 5$ and $\sin\left(\frac{3\pi}{2}(n+N_2)\right) = \sin\left(\frac{3\pi}{2}n\right)$ for $N_2 = 4$. If we take $N$ equal to the least common multiple of $N_1$ and $N_2$ we satisfy (1). Hence $N = 20$.

## Problem 2

(b) This is shown using the fact that $W_N^{nL}x[n] \overset{\text{DFT}}{\longleftrightarrow} X[x+L]$ and the linearity of the DFT.

(c) If $x[n]$ is even, i.e., $x[n] = x[N-n]$, then $X[k] = X[-k]$. Indeed,

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x[N-n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}k(N-n)} \\ &= \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}kn} \\ &= X[-k]. \end{aligned}$$

Thus,

$$\begin{aligned} Y[-k-1] &= X_1[-k] + X_2[-k-1] \\ &= X_1[k] + X_2[k+1]. \end{aligned}$$

(d) Let $X_1[0] = \sum_{n=0}^{N-1} x_1[n]$ and $X_2[0] = \sum_{n=0}^{N-1} x_2[n]$. The recursion is then

$$\begin{aligned} X_2[k+1] &= Y[N-(k+1)] - X_1[k] \\ X_1[k+1] &= Y[k] - X_2[k]. \end{aligned}$$

## Problem 3

(a) The period is $N = 20$, and we have

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{19} e^{-2n} e^{-j\frac{2\pi}{20}kn} \\ &= \frac{1 - e^{-(40+j2\pi k)}}{1 - e^{-\left(2+j\frac{2\pi}{20}k\right)}} \\ &= \frac{1 - e^{-40}}{e^{-(1+j\frac{\pi}{20}k)}\left(e^{1+j\frac{\pi}{20}k} - e^{-(1+j\frac{2\pi}{20}k)}\right)}, \qquad k = 0, 1, \ldots, 19\end{aligned}$$

(b) Here the period is $N = 2$, so

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{1} x[n] e^{-j\frac{2\pi}{N}kn} \\ &= 1 - e^{-j\pi k} \\ &= \begin{cases} 0, & k \text{ even} \\ 2, & k \text{ odd.} \end{cases}\end{aligned}$$

## Problem 4 (MATLAB AND FFT)

(b) Using $\sin x = (e^{jx} - e^{-jx})/2j$ and $N = 8$, we have

$$\begin{aligned}x[n] &= \frac{1}{2j}\left(e^{j\frac{2\pi n}{8}} - e^{-j\frac{2\pi n}{8}}\right) \\ &= \frac{1}{2j}\left(W_8^{-n} - W_8^{n}\right),\end{aligned}$$

so the DFT is given by (shift property)

$$X[k] = \frac{8}{2j}\left(\delta[k-1] - \delta[k-7]\right), \quad k = 0, \ldots, 7.$$

Computing the magnitude of the DFT in MATLAB results in the plot shown on Figure 1.

(Alternatively, if you have used $N = 16$, then the DFT is $X[k] = (16/2j)\left(\delta[k-2] - \delta[k-14]\right)$, $k = 0, \ldots, 15$.)

## Problem 5

(a) cf. Figure 2

(b) cf. Figure 3

(c) The DFTs of $x_1[k]$ and $x_2[k]$ are

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N-1$$

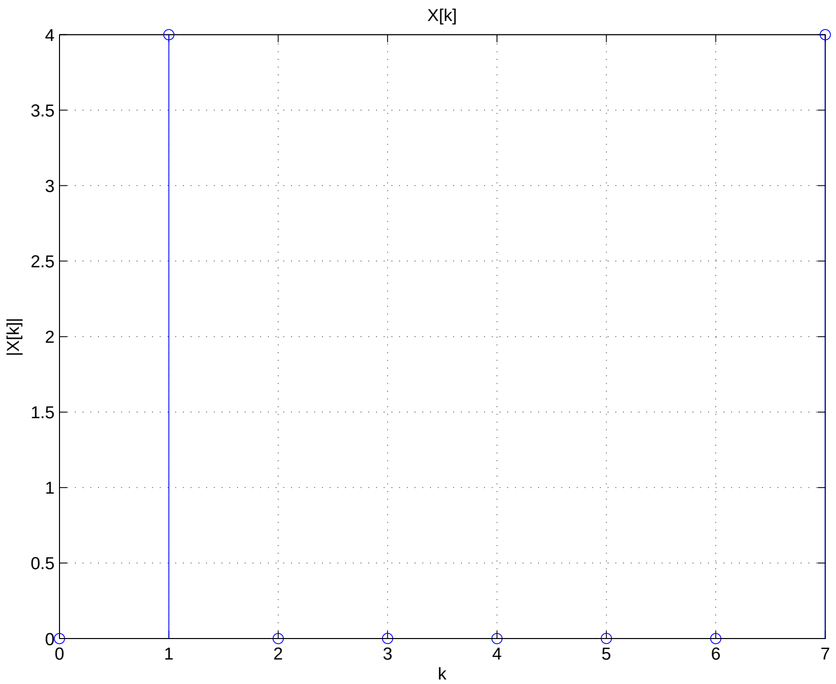


Figure 1: Magnitude plot for Problem 4.

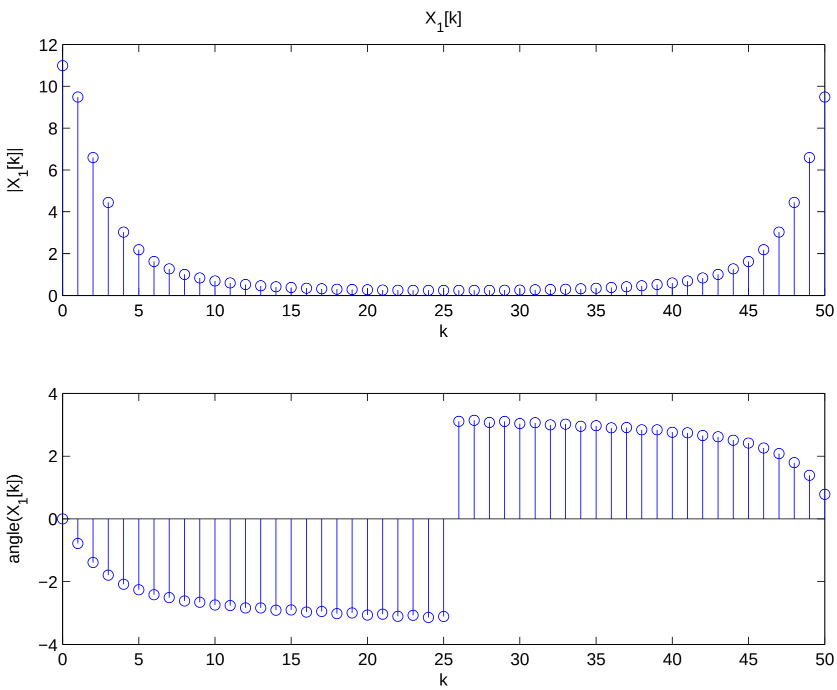


Figure 2: Magnitude and angle plot for Problem 5(a).

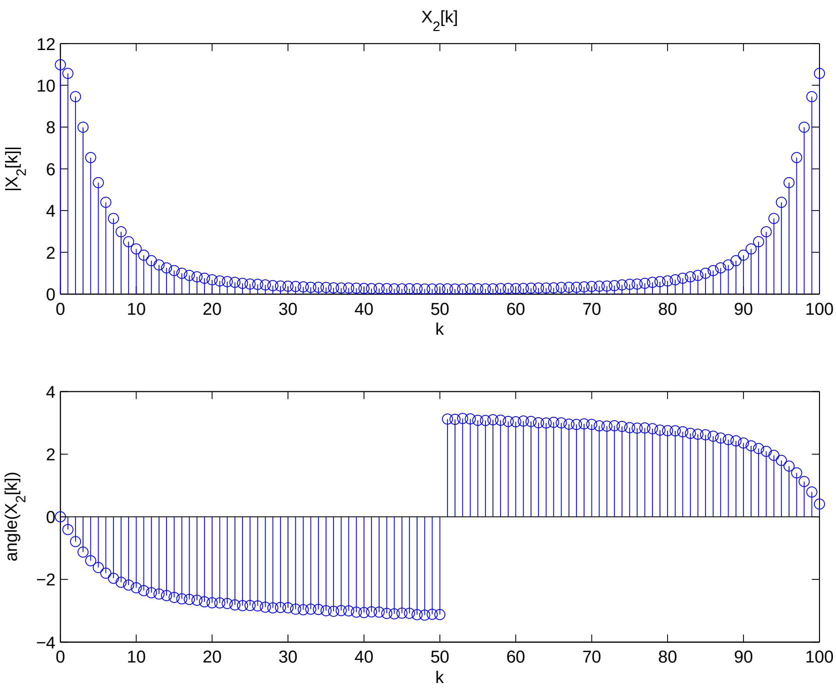


Figure 3: Magnitude and angle plot for Problem 5(b).

and

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{2\pi}{2N}kn}, \quad k = 0, \dots, 2N-1.$$

Writing

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{2N}2kn} = X_2[2k],$$

we see that $X_1[k]$ is a decimated version of $X_2[k]$, with a decimation factor of 2.

(d) cf. Figure 4

(e) We have

$$\begin{aligned} X_3[k] &= \sum_{n=0}^{2N-1} x_3[n] e^{-j\frac{2\pi}{2N}kn} \\ &= \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=N}^{2N-1} x_1[n-N] e^{-j\frac{2\pi}{2N}kn} \\ &= \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n'=0}^{N-1} x_1[n'] e^{-j\frac{2\pi}{2N}k(n'+N)} \\ &= \left(1 + e^{-j\frac{2\pi}{2N}kN}\right) \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{2N}kn} \\ &= \begin{cases} 2X_1[k/2], & k \text{ even} \\ 0, & k \text{ odd}, \end{cases} \end{aligned}$$

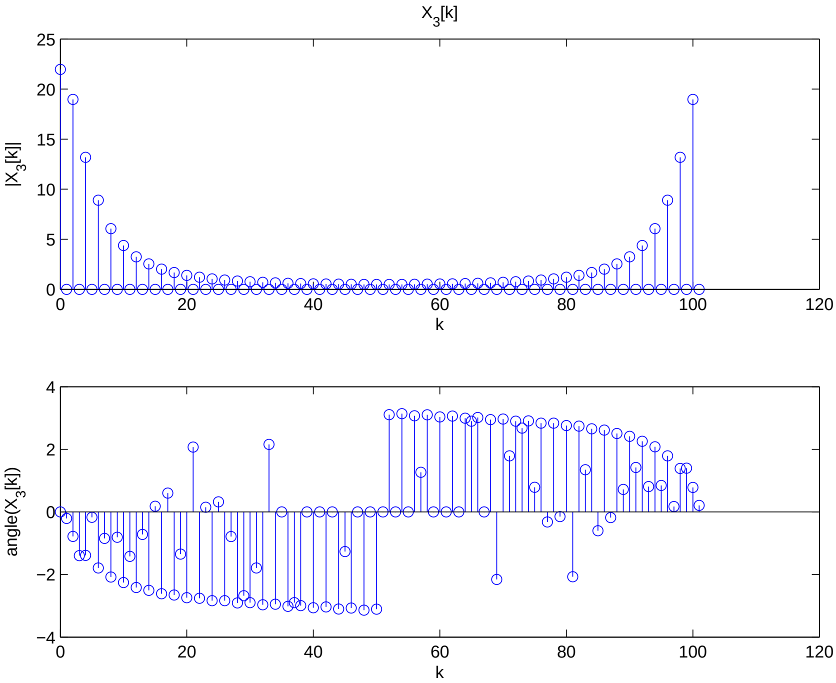


Figure 4: Magnitude and angle plot for Problem 5(d).

which explains the figure. (The non-zero angle values for odd $k$ are due to rounding errors).

## Problem 6 (DFT and DTFT)

(a) The following listing shows a possible way to compute the DTFT.

```
function X = dtft(x, w, n0)
% DTFT   Compute discrete-time Fourier transform
%    X = DTFT(x, w, n0) computes the discrete time Fourier transform of the
%    sequence x, evaluated at the frequencies indicated by w. Values of w
%    must be between 0 and 2 pi. n0 indicates the index of the first sample
%    of x; if unspecified it is assumed to be 0.

% Default arguments
if nargin < 3
    n0 = 0;
end

% Indices of the sequence
n = n0:n0 + length(x) - 1;

% Allocate output vector
X = zeros(1, length(w));

% Loop for summation
```

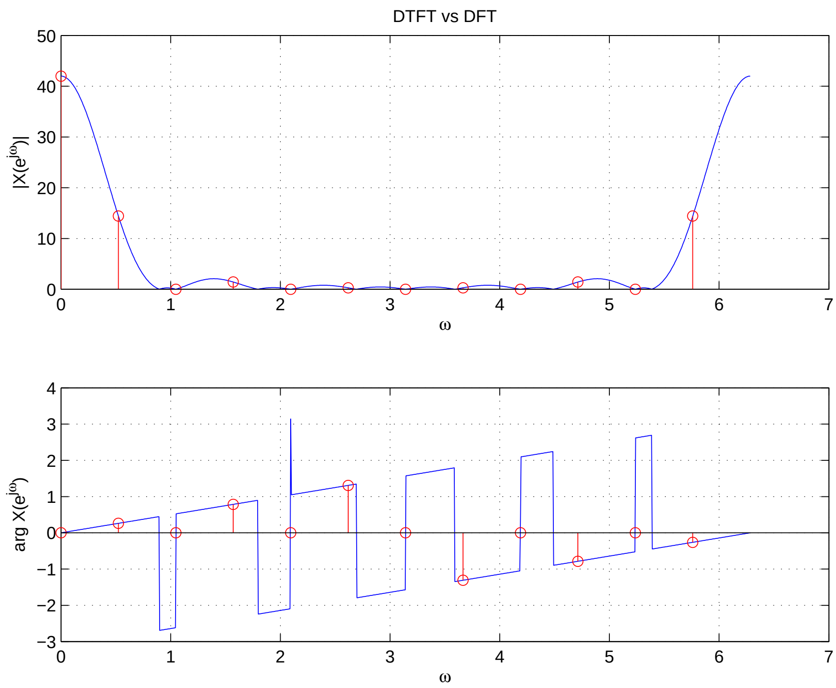


Figure 5: Plots for Problem 6. You can see that the DFT is a sampled version of the DTFT.

```
for k = 1:length(n)

    % Get current index
    l = n(k);

    % Add term to DTFT sum
    X = X + x(k) * exp(-j*w*l);
end
```

(b), (c) The resulting plots are shown on Figure 5.