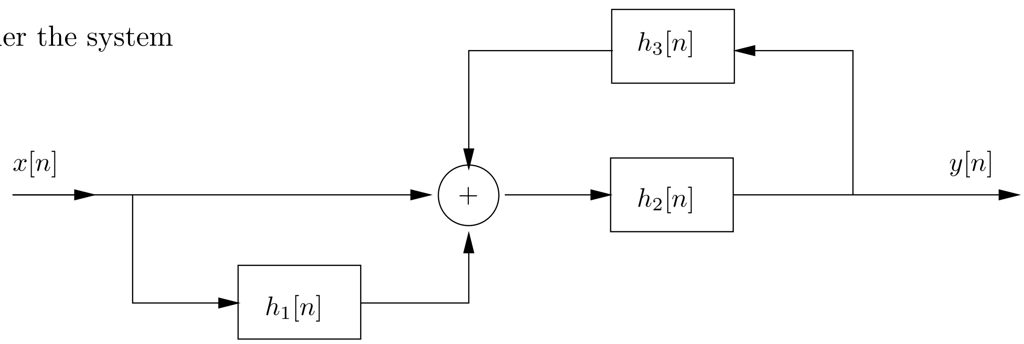
Homework Set # 5

Problem 1

Consider the system



with $h_1[n] = \beta\delta[n-1]$ and $h_2[n] = \alpha^n u[n]$ where $|\alpha| < 1$. **Through parts (a) to (e) assume $\mathbf{h_3[n] = 0}$.**

(a) Find the impulse response $h[n]$ of the overall system, *i.e.*, $h[n]$ such that

$$y[n] = h[n] * x[n].$$

(b) Find $H(z)$, the z-transform of the overall system and specify the ROC in terms of $|\alpha|$.

(c) Find the difference equation relating $y[n]$ to input $x[n]$.
Hint: Use the transfer function, $H(z)$.

(d) Is this system causal? Why?

(e) Is the system stable? Prove or disprove.

(f) If $h_1[n] = 2\delta[n-1]$, $h_2[n] = \frac{1}{2}\delta[n-1]$, and $h_3[n] = \frac{1}{3}\delta[n-1]$, find $h[n]$.

Problem 2

Suppose that we know that:

$$w[n] = \begin{cases} \frac{1}{n+1}h[n] & \text{, for } n > 0 \\ 0 & \text{, else} \end{cases}$$

(a) If $h[n]$ is a strictly causal sequence ($i.e.$, $h[0] = 0,\ n \leq 0$) then find $H(z)$ in terms of $W(z)$ and the corresponding ROC $\mathcal{R}_h$ in terms of $\mathcal{R}_w$, the ROC of $W(z)$.

(b) If $w[n] = a^n u[n-1]$, find $W(z)$, the z-transform of $w[n]$ and its corresponding ROC $\mathcal{R}_w$.

(c) For what value of a, does the DTFT of $w[n]$ exist?

(d) Find $h[n]$ corresponding to the $w[n]$ given in part (b). Does this correspond to a stable system? Note that your answer can depend on a.

(e) Suppose that $G(z)$ is known to be a system such that the relationship shown in Fig. 1 holds. Express $G(z)$ in terms of $H(z)$.

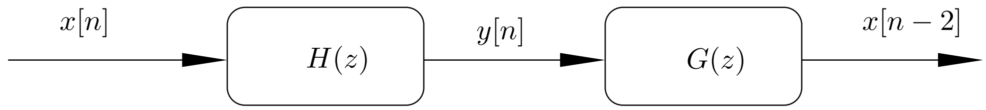


Figure 1: Relationship for problem 2(d)

(f) Given the $H(z)$ and the corresponding ROC found in (c), state if $G(z)$ can be causal and can it be stable.

Problem 3

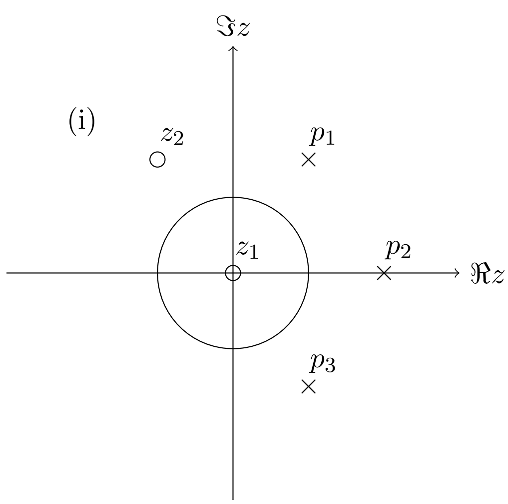
(a) Assume that we are given a system that has a rational system function $H(z)$. Below, you find four different plots of the poles and zeros of $H(z)$. For each situation, choose which one of the following assertions is true:

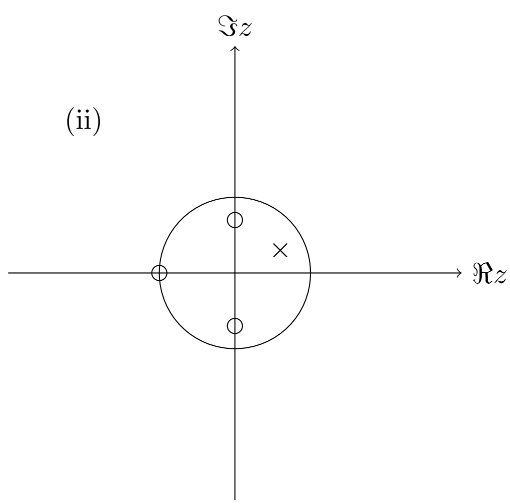
1. the system is stable if it is causal
2. the system is stable if it is anticausal
3. the system can only be stable if the impulse response is two-sided.

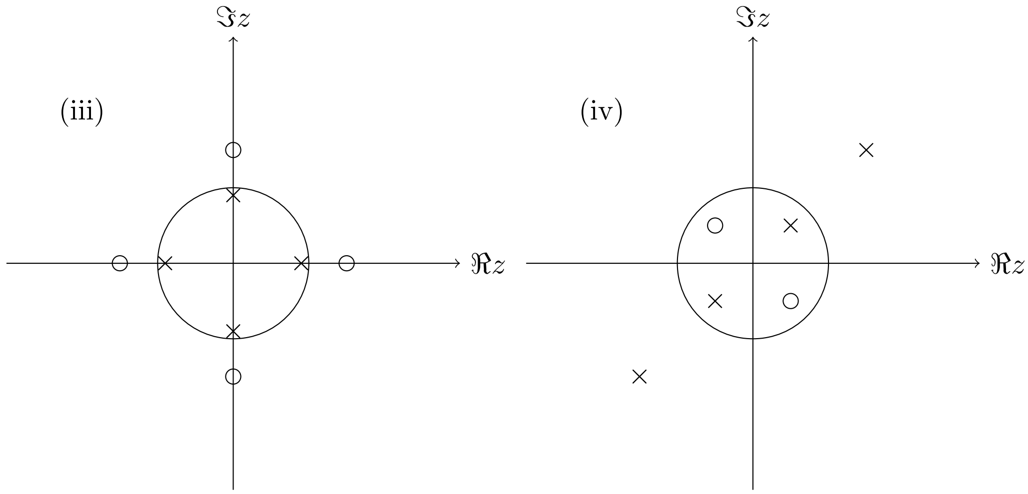
(b) For plot (i), we know that

$$H(z) = \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})(1 - p_3 z^{-1})}.$$

Give the system function $G(z)$ of the inverse system, *i.e.*, the system such that $h[n] * g[n] = \delta[n]$. For $G(z)$, determine which of the assertions 1. through 3. is true.







Problem 4

A linear time-invariant system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

(a) What are the zeros and poles of $H(z)$? Give a pole-zero plot.

(b) Find the partial fraction decomposition of $H(z)$, *i.e.*, rewrite $H(z)$ as $H(z) = \sum_k \frac{A_k}{1-d_k z^{-1}}$.

(c) Specify the ROC of $H(z)$ and determine $h_i[n]$ $(i = 1, 2)$ in the following two cases:

  (i) $h_1[n]$: the system is stable

  (ii) $h_2[n]$: the system is anticausal

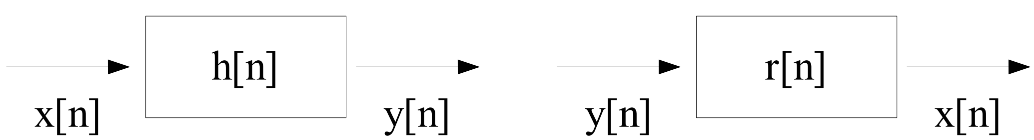
(d) For which of the sequences above can you compute the DTFT? Justify.
Hint: Do not compute DTFT, just state whether DTFT exists or not.

Problem 5

Consider the system characterized by the difference equation

$$y[n] = 0.5y[n-1] + x[n] - 3x[n-1], \qquad y[n] = 0 \text{ for } n < 0$$

i.e., we have zero initial conditions and we are interested in the evolution of the system for $n \geq 0$.



(a) System　　(b) Inverse system

Figure 2: The system and its inverse

(a) Provide an expression for $H(z)$, and explicitly state the ROC for $H(z)$.

(b) Show that you can write $H(z)$ as

$$H(z) = H_{min}(z)H_{ap}(z)$$

where $H_{min}(z)$ is a causal, minimum phase system and $H_{ap}(z)$ is an all-pass system. That is, all the poles and zeros of $H_{min}(z)$ are within the unit circle and

$$|H_{ap}(z)|_{z=e^{j\omega}} = 1.$$

Hint:

- An all-pass filter is a system which passes all the frequencies with the same gain, *i.e.*, $\left|H(e^{j\omega})\right| = 1$ for all ω. The general form for an all-pass filter is

$$H_{ap}(z) = \prod_{n=1}^{N} \frac{z^{-1} - d_n}{1 - d_n z^{-1}} \prod_{m=1} \frac{(z^{-1} - e_m^*)(z^{-1} - e_m)}{(1 - e_m z^{-1})(1 - e_m^* z^{-1})}$$

  where d_n's are the real poles and e_m's are the complex poles.
- For a stable and causal LTI system to be minimum phase, all its zeros must lie inside the unit circle.

(c) By inspecting $H(z)$, and its ROC, is it a BIBO stable system?

(d) Suppose we want to recover $x[n]$ from $y[n]$ (see Fig. 2(b)), find the filter $R(z)$ such that

$$X(z) = R(z)Y(z), \tag{1}$$

and explicitly state its ROC. Can $R(z)$ be a stable and causal system?

(e) Consider the system shown in Fig. 3. What are the amplitude and phase responses of $G(e^{j\omega})$?

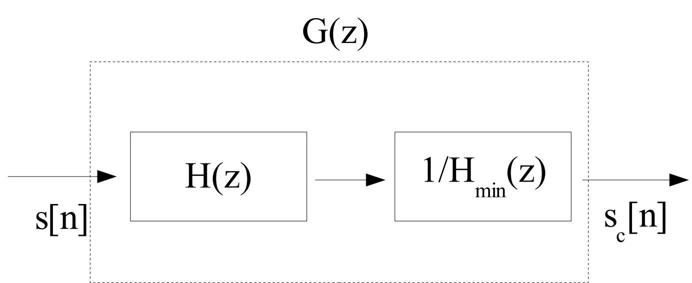


Figure 3: Distortion and compensation by a linear filter.