Homework Set # 3

Problem 1 (DTFT)

Let $u[n]$ be the unit step sequence, defined as $u[n] = 1$ for $n \geq 0$ and $u[n] = 0$ otherwise.

(a) Compute the DTFT of $x[n] = u[n-2] - u[n-4]$.

(b) Compute the DTFT of $h[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$.

(c) Compute the DTFT of $y[n] = x[n] * h[n] = \sum_k x[k]h[n-k]$.

(d) Compute the DTFT of $z[n] = y[n - n_0]$.

Problem 2 (DFS)

Let $\tilde{x}[n]$ and $\tilde{y}[n]$ be two periodic sequences defined as

$$\tilde{x}[n] = 1 + \cos\left(\frac{2\pi}{6}n\right) \quad \text{and} \quad \tilde{y}[n] = \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right). \tag{1}$$

(a) Compute the DFS coefficients of $\tilde{x}[n]$ and $\tilde{y}[n]$.

(b) Let $\tilde{u}[n]$ and $\tilde{v}[n]$ be two periodic sequences that both have period N. Furthermore, let

$$\tilde{w}[n] = \tilde{u}[n]\tilde{v}[n].$$

The DFS coefficients of $\tilde{w}[n]$, $\tilde{u}[n]$ and $\tilde{v}[n]$ are denoted by $\tilde{W}[k]$, $\tilde{U}[k]$ and $\tilde{V}[k]$ respectively, $k = 0, \ldots, N-1$.

Show that the following holds:

$$\tilde{W}[k] = \frac{1}{N}\sum_{l=0}^{N-1} \tilde{U}[l]\tilde{V}[k-l]. \tag{2}$$

(c) Let us define $\tilde{r}[n] = \tilde{W}[n]$. Let $\tilde{R}[k]$ be the DFS coefficients of $\tilde{r}[n]$. Express $\tilde{R}[k]$ in terms of $\tilde{u}[n]$ and $\tilde{v}[n]$.

(d) Let $\tilde{z}[n] = \tilde{x}[n]\tilde{y}[n]$, for $\tilde{x}[n]$ and $\tilde{y}[n]$ as given in Equation (1). Compute the DFS coefficients of $\tilde{z}[n]$.

Problem 3

Suppose that $x[n]$ is a *non-zero* sequence of length N, *i.e,*

$$x[n] = 0, \quad \text{for } n < 0 \text{ and } n > N-1,$$

and $x[n] \neq 0$ for some $n \in 0 \leq n \leq N-1$. Further, let us define

$$X[k] = \sum_n x[n] e^{-j\frac{2\pi}{M}kn}, \quad \text{for } k = 0, \ldots, M-1.$$

(a) Prove or disprove the following:

1. It is possible that $X[k] = 0$ for *all* $k = 0, \ldots, M-1$ if $M \geq N$. If so, give an example. If not, prove that it is not possible.
2. It is possible that $X[k] = 0$ for *all* $k = 0, \ldots, M-1$ if $M < N$. If so, give an example. If not, prove that it is not possible.

(b) Now, set $M = 2N$ and define:

$$X_1[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad \text{for } k = 0, \ldots, N-1 \tag{3}$$

$$X_2[k] = \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}kn}, \quad \text{for } k = 0, \ldots, M-1. \tag{4}$$

What is the relationship between $X_1[k]$ and $X_2[k]$?

Problem 4

Let $\mathbf{e}_i$, $i = 1, \ldots, 8$, be a vector of length eight, with all zero elements, except an one at the i-th position, *i.e.,*

$$\mathbf{e}_i = [0, \ldots, 0, \underbrace{1}_{i\text{−th position}}, 0, \ldots, 0]^t, \quad i = 1, \ldots, 8.$$

Then $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$ is an orthonormal basis for $\mathbb{R}^8$. Consider the following set of vectors in $\mathbb{R}^8$,

$$\tilde{\mathbf{h}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}_E,$$

$$\tilde{\mathbf{h}}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_6 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_7 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}_E, \quad \tilde{\mathbf{h}}_8 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}_E,$$

where subscript E shows that they are defined in E-basis.

(a) Clarify whether $\tilde{H} = \{\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_8\}$ forms an orthonormal basis for $\mathbb{R}^8$ or not. If so, define $H = \tilde{H}$, otherwise modify $\tilde{\mathbf{h}}_i$'s to obtain an orthonormal basis $H = \{\mathbf{h}_1, \dots, \mathbf{h}_8\}$.

(b) Let $\mathbf{a}_H = [2, 0, -1, 4, -3, 1, 1, 0]$ be a vector in basis H. Compute the representation of $\mathbf{a}_H$ in basis E. *(You can use MATLAB for computation in all parts of this problem.)*

(c) Let we want to measure some vectors in $\mathbb{R}^6$ which are the results of some experiment. The device we use to do this measurement, measures the vectors of length 8, and shows them in H-basis. One way to adapt the device to our purpose is to consider our vectors as length 8, vectors wherein the last two components are zero in E-basis, *e.g.,* vector $[1, -3, 2, 0, 1, 4]_E$ is considered as $[1, -3, 2, 0, 1, 4, 0, 0]_E$ and then showed in H-basis by the device. Unfortunately the vectors measured by the device are not accurate and corrupted by adding by some error vectors.

Assume $\mathbf{b}_E = [b_1, b_2, \dots, b_6, 0, 0]$ be the actual vector in E-basis we are intersted in. Let $\mathbf{b}_H = [0, -4, -6, 2, 0, 12, -2, 6]$ be the vector measured by the device. Find $\hat{\mathbf{b}}_E$, the best estimation for $\mathbf{b}_E$ which minimizes $|\mathbf{b} - \hat{\mathbf{b}}|^2$.

(d) Assume we are given a set of vectors of length 8, and we want to store them. Because of lack of memory we can only store five elements of each vector, additional to the position of the stored elements, such that one can reconstruct the vector by adding zeros at the missing positions. Let $\mathbf{x}_E = [5, 2, -7, 8, 0, 10, 3, -2]$ be the vector to be stored, and $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$, be the reconstructed vectors from the following elements of $\mathbf{x}_E$.

$$\begin{aligned}
\hat{\mathbf{x}}_1 &: \text{ reconstructed from } x \text{ at positions } (1,2,3,4,5) \\
\hat{\mathbf{x}}_2 &: \text{ reconstructed from } x \text{ at positions } (1,3,4,6,7) \\
\hat{\mathbf{x}}_3 &: \text{ reconstructed from } x \text{ at positions } (1,2,4,6,7).
\end{aligned}$$

Compute the value of error, $|\mathbf{x} - \hat{\mathbf{x}}_j|^2$ for $j = 1, 2, 3$, and specify which one has the smallest reconstruction error.

(e) Having the result of part (d) guess the optimal storage method, *i.e.,* for a given n and k, determine the k positions of an n-vector $\mathbf{x}$ to store in order to minimize the reconstruction error, $|\mathbf{x} - \hat{\mathbf{x}}|$. Proof your guess.