Chapter 4

Signals and Hilbert Spaces

In the 17th century, algebra and geometry started to interact in a fruitful synergy which continues to the present day. Descartes's original idea of translating geometric constructs into algebraic form spurred a new line of attack in mathematics; soon, a series of astonishing results was produced for a number of problems which had long defied geometrical solutions (such as, famously, the trisection of the angle). It also spearheaded the notion of vector space, in which a geometrical point could be represented as an n-tuple of coordinates; this, in turn, readily evolved into the theory of linear algebra. Later, the concept proved useful in the opposite direction: many algebraic problems could benefit from our innate geometrical intuition once they were cast in vector form; from the easy three-dimensional visualization of concepts such as distance and orthogonality, more complex algebraic constructs could be brought within the realm of intuition. The final leap of imagination came with the realization that the concept of vector space could be applied to much more abstract entities such as infinite-dimensional objects and functions. In so doing, however, spatial intuition could be of limited help and so the notion of vector space had to be formalized in much more rigorous terms; we will see that the definition of Hilbert space is one such formalization.

Most of the signal processing theory which we will study in the course can be usefully cast in terms of vector notation and the advantages of this approach are exactly what we just delineated before. First of all, all the standard machinery of linear algebra becomes immediately available and applicable; this greatly simplifies the formalism used in the mathematical proofs which will follow and, at the same time, it fosters a good intuition with respect to the underlying principles which are being put in place. Furthermore, the vector notation creates a frame of thought which seamlessly links the more abstract results involving infinite sequences to the algorithmic reality involving finite-length signals. Finally, on the practical side, vector notation is the standard paradigm for numerical analysis packages such as Matlab; signal processing algorithms expressed in vector notation translate to working code with very little effort.

In Chapter 2 we established the basic notation for the different classes of discrete-time signals which we will encounter time and again in the rest of the course and we hinted at the fact that a tight correspondence can be established between the concept of signal and that of vector space. In this chapter we will pursue this link further, firstly by reviewing the familiar Euclidean spaces in finite dimensions and then by extending the concept of basic vector spaces to infinite-dimensional Hilbert spaces.

4.1 A Quick Review of Euclidean Geometry

Euclidean geometry is a straightforward formalization of our spatial sensory experience; hence its cornerstone role in developing a basic intuition for vector spaces. Everybody is (or should be) familiar with Euclidean geometry and the natural "physical" spaces like $\mathbb{R}^2$ (the plane) and $\mathbb{R}^3$ (the three-dimensional space). The notion of *distance* is clear, *orthogonality* is intuitive and maps to the idea of a "right angle". Even a more abstract concept such as that of *basis* is rather easy to think of (the standard coordinate concepts of latitude, longitude and height, which correspond to the three orthogonal axes in $\mathbb{R}^3$). Unfortunately, immediate spatial intuition fails us for higher dimensions (i.e. for $\mathbb{R}^N$ with $N > 3$), yet the basic concepts introduced for $\mathbb{R}^3$ generalize easily to $\mathbb{R}^N$ so that it is easier to state such concepts for the higher-dimensional case and specialize them with examples for $N = 2$ or $N = 3$. These notions ultimately will be generalized even further to more abstract types of vector spaces. For the moment, let us review the properties of $\mathbb{R}^N$, the N-dimensional Euclidean space.

Vectors and Notation. A point in $\mathbb{R}^N$ is specified by an N-tuple of coordinates[1]:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = [x_0 \; x_1 \; \dots \; x_{N-1}]^T$$

where $x_i \in \mathbb{R},\ i = 0, 1, \dots, N-1$. We call this set of coordinates a *vector* and the N-tuple will be denoted synthetically by the symbol $\mathbf{x}$; coordinates are usually expressed with

[1] N-dimensional vectors are by default *column* vectors.

respect to a "standard" orthonormal basis[2]. The vector $\mathbf{0} = [0\ 0\ \ldots\ 0]^T$, i.e. the null vector, is considered the origin of the coordinate system.

The generic n-th element in vector $\mathbf{x}$ is indicated by the subscript x_n. In the following we will often consider a *set* of M arbitrarily chosen vectors in $\mathbb{R}^N$ and this set will be indicated by the notation $\{\mathbf{x}^{(k)}\}_{k=0\ldots M-1}$. Each vector in the set is indexed by the superscript $\cdot^{(k)}$. The n-th element of the k-th vector in the set is indicated by the notation $x_n^{(k)}$

Inner Product. The inner product between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ is defined as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x_n y_n. \tag{4.1}$$

We say that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, or $\mathbf{x} \perp \mathbf{y}$, when the inner product is zero:

$$\mathbf{x} \perp \mathbf{y} \iff \langle \mathbf{x}, \mathbf{y} \rangle = 0. \tag{4.2}$$

Norm. The norm of a vector is defined in terms of the inner product as

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{n=0}^{N-1} x_n^2} = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}. \tag{4.3}$$

It is easy to visualize geometrically that the norm of a vector corresponds to its length, i.e. to the distance between the origin and the point identified by the vector's coordinates. A remarkable property linking the inner product and the norm is the Cauchy-Schwarz inequality (whose proof is a little tricky); given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we *always* have:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

Distance. The concept of norm is used to introduce the notion of Euclidean *distance* between two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{\sum_{n=0}^{N-1} (x_n - y_n)^2}. \tag{4.4}$$

From this, we can easily derive the Pythagorean theorem for N dimensions: if two vectors are orthogonal, $\mathbf{x} \perp \mathbf{y}$, and we consider the sum vector $\boldsymbol{z} = \mathbf{x} + \mathbf{y}$, we have:

$$\|\boldsymbol{z}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2. \tag{4.5}$$

[2] The concept of basis will be defined more precisely later on; for the time being, consider a standard set of orthogonal axes.

The above properties are graphically shown in Figure 4.1 for $\mathbb{R}^2$.

Bases. Consider a set of M arbitrarily chosen vectors in $\mathbb{R}^N$: $\{\mathbf{x}^{(k)}\}_{k=0\ldots M-1}$. Given such a set, a key question is that of completeness: can *any* vector in $\mathbb{R}^N$ be written as a linear combination of vectors from the set? In other words, we ask ourselves whether for any $\mathbf{z} \in \mathbb{R}^N$ we can find a set of M coefficients $\alpha_k \in \mathbb{R}$ such that $\mathbf{z}$ can be expressed as:

$$\mathbf{z} = \sum_{k=0}^{M-1} \alpha_k \mathbf{x}^{(k)}. \tag{4.6}$$

Clearly, M needs to be greater or equal to N, but what conditions does a set of vectors $\{\mathbf{x}^{(k)}\}_{k=0\ldots M-1}$ need to satisfy so that (4.6) holds for any $\mathbf{z} \in \mathbb{R}^N$? There needs to be a set of M vectors that *span* $\mathbb{R}^N$, and it can be shown that this is equivalent to saying that the set must contain at least N *linearly independent* vectors. In turn, N vectors $\{\mathbf{y}^{(k)}\}_{k=0\ldots N-1}$ are linearly independent if the equation

$$\sum_{k=0}^{N-1} \beta_k \mathbf{y}^{(k)} = 0 \tag{4.7}$$

is satisfied only when all the β_k's are zero. A set of N linearly independent vectors for $\mathbb{R}^N$ is called a *basis* and, amongst bases, the ones with mutually orthogonal vectors of norm equal to one are called *orthonormal bases.* For an orthonormal basis $\{\mathbf{y}^{(k)}\}$ we therefore have:

$$\langle \mathbf{y}^{(k)}, \mathbf{y}^{(h)} \rangle = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Figure 4.2 reviews the above concepts in low dimensions.

The standard orthonormal basis for $\mathbb{R}^N$ is the *canonical basis* $\{\boldsymbol{\delta}^{(k)}\}_{k=0\ldots N-1}$ with

$$\delta_n^{(k)} = \delta[n-k] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

The orthonormality of such a set is immediately apparent. Another important orthonormal basis for $\mathbb{R}^N$ is the normalized *Fourier basis* $\{\mathbf{w}^{(k)}\}_{k=0\ldots N-1}$ for which:

$$\mathbf{w}_n^{(k)} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}nk};$$

the orthonormality proof for the basis is left as an exercise.

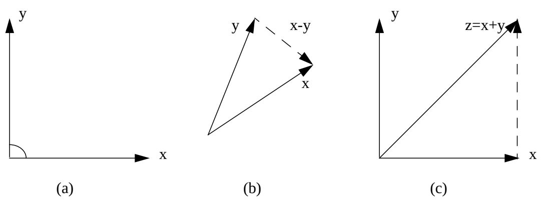


Figure 4.1: Elementary properties of vectors in $\mathbb{R}^2$. (a) Orthogonality of two vectors $\mathbf{x}$ and $\mathbf{y}$. (b) Difference vector $\mathbf{x} - \mathbf{y}$ and distance between $\mathbf{x}$ and $\mathbf{y}$. (c) Sum of two orthogonal vectors $z = \mathbf{x} + \mathbf{y}$, and Pythagorean theorem.

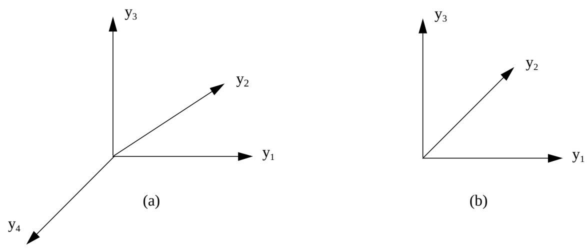


Figure 4.2: Linear independence and bases. (a) $(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)})$ are coplanar in $\mathbb{R}^3$, and so do not form a basis. $\mathbf{y}^{(4)}$ and any two of $\{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)}\}$ are linearly independent. (b) Any two of $\{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)}\}$ form a basis, and $\{\mathbf{y}^{(1)}, \mathbf{y}^{(3)}\}$ form an orthogonal basis.

4.2 From Vector Spaces to Hilbert Spaces

The purpose of the previous review was to briefly review the elementary notions and spatial intuitions of Euclidean geometry. A thorough study of vectors in $\mathbb{R}^N$ and $\mathbb{C}^N$ is the subject of linear algebra; yet, the idea of vectors, orthogonality and bases is much more general, the basic ingredients being an inner product and the use of a square norm as in (4.3).

While the analogy between vectors in $\mathbb{C}^N$ and length-N signals is readily apparent, the question now hinges on how we are to proceed in order to generalize the above concepts to the class of infinite sequences. Intuitively, for instance, we can let N grow to infinity and obtain $\mathbb{C}^\infty$ as the Euclidean space for infinite sequences; in this case, however, much care must be exercised with expressions such as (4.1) and (4.3) which can diverge for sequences as simple as $x[n] = 1$ for all n. In fact, the proper generalization of $\mathbb{C}^N$ to an infinite number of dimensions is in the form of a particular vector space called *Hilbert space*; the structure of this kind of vector space imposes a set of constraints on its elements so that divergence problems such as the one we just mentioned no longer bother us. When we embed infinite sequences into a Hilbert space, these constraints translate to the condition that the corresponding signals have finite energy — which is a mild and reasonable requirement.

Finally, it is important to remember the notion of Hilbert space is applicable to much more general vector spaces than $\mathbb{C}^N$; for instance, we can easily consider spaces of functions over an interval or over the real line. This generality is actually the cornerstone of a branch of mathematics called *functional analysis*. While we will not tread very far into these kind of generalizations, we will certainly point out a few of them along the way. The space of square integrable functions, for instance, will turn out to be a marvelous tool a few chapters from now when, finally, the link between continuous- and discrete-time signals will be explored in detail.

4.2.1 The Recipe for Hilbert Spaces

A word of caution: we are now starting to operate in a world of complete abstraction. Here a vector is an entity *per se* and, while analogies and examples in terms of Euclidean geometry can be useful visually, they are by no means exhaustive. In other words: vectors are no longer just N-tuples of numbers; they can be anything. This said, a Hilbert space can be defined in incremental steps starting from a general notion of vector space and by supplementing this space with two additional features: the existence of an inner product and the property of completeness.

Vector Space. Consider a set of vectors V and a set of scalars S (which can be either $\mathbb{R}$

or $\mathbb{C}$ for our purposes). A vector space $H(V, S)$ is completely defined by the existence of a vector addition operation and a scalar multiplication operation which satisfy the following properties for any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \in V$ and any $\alpha, \beta \in S$:

- Addition is commutative:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} \tag{4.9}$$

- Addition is associative:

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) \tag{4.10}$$

- Scalar multiplication is distributive:

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} \tag{4.11}$$

$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x} \tag{4.12}$$

$$\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x} \tag{4.13}$$

- There exists a null vector $\mathbf{0}$ in V which is the additive identity so that $\forall\, \mathbf{x} \in V$:

$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x} \tag{4.14}$$

- $\forall\, \mathbf{x} \in V$ there exists in V an additive inverse $-\mathbf{x}$ such that:

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0} \tag{4.15}$$

- There exists an identity element "1" for *scalar* multiplication so that $\forall\, \mathbf{x} \in V$:

$$1 \cdot \mathbf{x} = \mathbf{x} \cdot 1 = \mathbf{x}. \tag{4.16}$$

Inner Product Space. What we have so far is the simplest type of vector space; the next ingredient which we will consider is the *inner product* which is essential to build a notion of *distance* between elements in a vector space. A vector space with an inner product is called an inner product space. An inner product for $H(V, S)$ is a function from $V \times V$ to S which satisfies the following properties for any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \in V$:

- It is *distributive* with respect to vector addition:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \tag{4.17}$$

- It possesses the *scaling property* with respect to scalar multiplication[3]:

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle \tag{4.18}$$

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle \tag{4.19}$$

- It is commutative within complex conjugation:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^* \tag{4.20}$$

- The self-product is real and positive:

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \tag{4.21}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle \in \mathbb{R}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Longleftrightarrow \mathbf{x} = \mathbf{0}. \tag{4.22}$$

From this definition of the inner product a series of additional definitions and properties can be derived: first of all, orthogonality between two vectors is defined with respect to the inner product, and we will say that non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, or $\mathbf{x} \perp \mathbf{y}$, if and only if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0. \tag{4.23}$$

From the definition of an inner product we can define the *norm* of a vector as:

$$||\mathbf{x}|| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}. \tag{4.24}$$

In turn, the norm satisfies the *Cauchy-Schwartz inequality*:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|, \tag{4.25}$$

with strict equality if and only if $\mathbf{x} = \alpha \mathbf{y}$.

[3]Note that in our notation, the left operand is conjugated.

Proof: If $\mathbf{y} = \mathbf{0}$, then (4.25) holds since $\langle \mathbf{x}, \mathbf{0} \rangle = 0$.
If $\mathbf{y} \neq \mathbf{0}$, then for every scalar α we have

$$0 \leq \|\mathbf{x} - \alpha\mathbf{y}\|^2 = \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \alpha \langle \mathbf{x}, \mathbf{y} \rangle - \alpha^* \left[\langle \mathbf{y}, \mathbf{x} \rangle - \alpha \langle \mathbf{y}, \mathbf{y} \rangle \right].$$

If we choose $\alpha = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$ then we have

$$0 \leq \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle = \frac{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle - |\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle},$$

or

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|,$$

with equality iff $\mathbf{x} = \alpha\mathbf{y}$. ■

The norm also satisfies the *triangle inequality*:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \tag{4.26}$$

with strict equality if and only if $\mathbf{x} = \alpha\mathbf{y}$ and $\alpha \in \mathbb{R}^+$.

Proof:

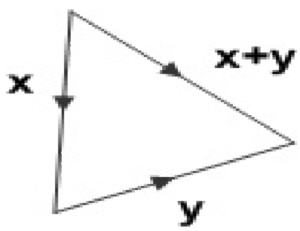
$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \|\mathbf{y}\|^2.$$

Since for any complex number u, $\mathrm{Re}(u) \leq |u|$ and $\mathrm{Im}(u) \leq |u|$, we have $\langle \mathbf{x}, \mathbf{y} \rangle \leq |\langle \mathbf{x}, \mathbf{y} \rangle|$, and $\langle \mathbf{y}, \mathbf{x} \rangle \leq |\langle \mathbf{x}, \mathbf{y} \rangle|$.
Hence

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Taking square roots we get

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|. \quad ■$$



For orthogonal vectors, the triangle inequality becomes the famous Pythagorean Theorem:

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \quad \text{for } \mathbf{x} \perp \mathbf{y}. \tag{4.27}$$

Hilbert Space. A vector space $H(V, S)$ equipped with an inner product is called an inner product space. To obtain a Hilbert space, we need completeness. This is a slightly more technical notion, which essentially implies that convergent sequences of vectors in V have a limit that is also in V. To gain intuition, think of the set of rational numbers $\mathbb{Q}$ versus the set of real numbers $\mathbb{R}$. The set of rational numbers is incomplete, because there are convergent sequences in $\mathbb{Q}$ which converge to irrational numbers. The set of real numbers contains these irrational numbers, and is in that sense the completion of $\mathbb{Q}$. Completeness is usually hard to prove in the case of infinite-dimensional spaces; in the following it will be tacitly assumed and the interested reader can easily find the relevant proofs in advanced analysis textbooks. As a last technicality, we will also only consider *separate* Hilbert spaces, which are the ones that admit orthonormal bases.

4.2.2 Examples of Hilbert Spaces

Finite Euclidean Spaces. The vector space $\mathbb{C}^N$, with the "natural" definition for the sum of two vectors $\mathbf{z} = \mathbf{x} + \mathbf{y}$ as:

$$z_n = x_n + y_n \tag{4.28}$$

and the definition of the inner product as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x_n^* y_n \tag{4.29}$$

is a Hilbert space.

Polynomial Functions. An example of "functional" Hilbert space is the vector space $\mathbb{P}_N([0,1])$ of polynomial functions on the interval $[0,1]$ with maximum degree N. It is a good exercise to show that $\mathbb{P}_\infty([0,1])$ is not complete: consider for instance the sequence of polynomials

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

this series converges as $p_n(x) \to e^x \notin \mathbb{P}_\infty([0,1])$.

Square Summable Functions. Another interesting example of functional Hilbert space is the space of *square integrable functions* over a finite interval. For instance, $L_2([-\pi,\pi])$ is the space of real or complex functions on the interval $[-\pi,\pi]$ which have finite norm. The inner product over $L_2([-\pi,\pi])$ is defined as:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f^*(t)g(t)dt, \tag{4.30}$$

so that the norm of $f(t)$ is:

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} |f(t)|^2 dt}. \tag{4.31}$$

For $f(t)$ to belong to $L_2([-\pi,\pi])$ it must have $\|f\| < \infty$.

4.2.3 Inner Products and Distances

The inner product is a fundamental tool in a vector space since it allows us to introduce a notion of *distance* between vectors. The key intuition about this is a typical instance in which a geometric construct helps us generalize a basic idea to much more abstract scenarios. Indeed, take the simple Euclidean space $\mathbb{R}^N$ and a given vector $\mathbf{x}$; for any vector $\mathbf{y} \in \mathbb{R}^N$ the inner product $\langle \mathbf{x}, \mathbf{y}\rangle$ is the measure of the *orthogonal projection* of $\mathbf{y}$ over $\mathbf{x}$. We know that the orthogonal projection defines the point on $\mathbf{x}$ which is closest to $\mathbf{y}$ and therefore it indicates "how well" we can approximate $\mathbf{y}$ by a simple scaling of $\mathbf{x}$. To see this, just note that

$$\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$$

where θ is the angle between the two vectors (you can work out the expression in $\mathbb{R}^2$ to easily convince you of this; the result generalizes to any other dimension). Clearly, if the vectors are orthogonal, the cosine is zero and no approximation will be possible. Since the inner product is dependent on the angular separation between the vectors, it represents a first rough measure of similarity between $\mathbf{x}$ and $\mathbf{y}$; in broad terms, it provides a measure of the difference in *shape* between vectors.

In the context of signal processing, this is particularly relevant since the difference in "shape" between signals is what we are interested in most of the time. As we stated several times before, discrete-time signals *are* vectors; the computation of their inner product will assume different names according to the processing context we find ourselves in: it will be called *filtering*, when we are trying to approximate or modify a signal; or it will be called *correlation* when we are trying to detect one particular signal amongst many. Yet, in all cases, it will still be an inner product, i.e. a *qualitative* measure of similarity between vectors. In particular, the concept of orthogonality between signals implies that the signals are perfectly distinguishable or, in other words, that their shape is completely different.

The need for a *quantitative* measure of similarity in some applications calls for the introduction of the Euclidean distance, which is derived from the inner product as:

$$d(\mathbf{x},\mathbf{y}) = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle^{1/2} = \|\mathbf{x}-\mathbf{y}\|. \tag{4.32}$$

In particular, for $\mathbb{C}^N$ the Euclidean distance is defined as:

$$d(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{n=0}^{N-1} |x_n - y_n|^2}; \tag{4.33}$$

whereas for $L_2([-\pi,\pi])$ we have:

$$d(\mathbf{x},\mathbf{y}) = \sqrt{\int_{-\pi}^{\pi} |x(t) - y(t)|^2 dt}. \tag{4.34}$$

In the practice of signal processing, the Euclidean distance is referred to as the *root mean square error*[4]; this is a global, quantitative goodness-of-fit measure when trying to approximate signal $\mathbf{y}$ with $\mathbf{x}$.

Incidentally, there are other types of distance measures which do not rely on a notion of inner product; for example in $\mathbb{C}^N$ we could define

$$d(\mathbf{x},\mathbf{y}) = \max_{0\leq n<N} |\mathbf{x}_n - \mathbf{y}_n|. \tag{4.35}$$

This distance is based on the supremum norm and is usually indicated by $\|\mathbf{x}-\mathbf{y}\|_\infty$; however, it can be shown that there is no inner product from which this norm can be derived and therefore no Hilbert space can be constructed where $\|\cdot\|_\infty$ is the natural norm. Nonetheless, this norm will reappear later, in the context of optimal filter design.

4.3 Subspaces, Bases, and Projections

Now that we have defined the properties of Hilbert spaces, it is only natural to start looking at the consequent inner *structure* of such a space. The best way to do so is by introducing the concept of *basis*. You can think of a basis as the “skeleton” of a vector space, a structure which holds everything together; yet, this skeleton is flexible and we can twist it, stretch it and rotate it in order to highlight some particular structure of the space

[4]Almost always, the square distance is considered instead; its name is then the *mean square error*, or MSE

and bring out the particular information we are interested in. All this is accomplished by a linear transformation called a *change of basis*; for instance, the Fourier transform is an instance of basis change.

Sometimes we will be interested in exploring more in detail a specific subset of a given vector space; this is accomplished via the concept of *subspace*. A subspace is, as the name implies, a restricted region of the global space with the additional properties that it is *closed* under the usual vector operations. This implies that, once in a subspace, we can operate freely without ever leaving its confines; just like a full-fledged space, a subspace has its own skeleton (i.e. the basis) and, again, we can exploit the properties of this basis to highlight the features we are interested in.

4.3.1 Definitions

Assume $H(V, S)$ is a Hilbert space, with V a vector space and S a set of scalars (i.e. $\mathbb{C}$).

Subspace. A subspace of V is defined as a subset $P \subseteq V$ that satisfies the following properties:

- Closure under addition, i.e.

$$\forall\, \mathbf{x}, \mathbf{y} \in P \Rightarrow \mathbf{x} + \mathbf{y} \in P \tag{4.36}$$

- Closure under scalar multiplication, i.e.

$$\forall\, \mathbf{x} \in P, \forall\, \alpha \in S \Rightarrow \alpha\mathbf{x} \in P. \tag{4.37}$$

Clearly, V is a subspace of itself.

Span. Given an arbitrary set of M vectors $W = \{\mathbf{x}^{(m)}\}_{m=0,1,\ldots,M-1}$, the *span* of these vector is defined as:

$$\mathrm{span}(W) = \left\{ \sum_{m=0}^{M-1} \alpha_m \mathbf{x}^{(m)} \right\}, \quad \alpha_m \in S \tag{4.38}$$

i.e. the span of W is the set of all possible linear combinations of the vectors in W. The set of vectors W is called *linearly independent* if the following holds:

$$\sum_{m=0}^{M-1} \alpha_m \mathbf{x}^{(m)} = 0 \quad \Longleftrightarrow \quad \alpha_m = 0 \text{ for } m = 0, 1, \ldots, M-1. \tag{4.39}$$

Basis. A set of K vectors $W = \{\mathbf{x}^{(k)}\}_{k=0,1,\ldots,K-1}$ from a subspace P is a *basis* for that subspace if:

- The set W is linearly independent.
- Its span covers P, i.e. $\text{span}(W) = P$.

The last statement affirms that any $\mathbf{y} \in P$ can be written as a linear combination of $\{\mathbf{x}^{(k)}\}_{k=0,1,\ldots,K-1}$ or that, for all $\mathbf{y} \in P$, there exist K coefficients α_k such that

$$\mathbf{y} = \sum_{k=0}^{K-1} \alpha_k \mathbf{x}^{(k)}; \tag{4.40}$$

this is equivalently expressed by saying that the set W is *complete* in P.

Orthogonal/Orthonormal Basis. An orthonormal basis for a subspace P is a set of K basis vectors $W = \{\mathbf{x}^{(k)}\}_{k=0,1,\ldots,K-1}$ for which:

$$\langle \mathbf{x}^{(i)}, \mathbf{x}^{(j)} \rangle = \delta[i-j] \quad 0 \leq i, j < K \tag{4.41}$$

which means orthogonality across vectors and unit norm. Sometimes, we can have that the set of vectors is orthogonal but not normal (i.e. the norm of the vectors is not unitary). This is hardly a problem provided that we remember to include the appropriate normalization factors in the analysis and/or synthesis formulas. Alternatively, an orthogonal set of vectors can be normalized via the Gram-Schmidt procedure, which you can find in any linear algebra textbook.

Among all bases, *orthonormal bases* are the most "beautiful" in some sense because of their structure and their properties. One of the most important properties for finite-dimensional spaces is the following:

- A set of N orthogonal vectors in an N-dimensional subspace is a basis for the subspace.

In other words, in finite dimensions, once we find a full set of orthogonal vectors we are sure that the set spans the space.

4.3.2 Properties of Orthonormal Bases

Let $W = \{\mathbf{x}^{(k)}\}_{k=0,1,\ldots,K-1}$ be an orthonormal basis for a (sub)space P. Then the following properties hold (all of which are easily verified):

Analysis Formula. The coefficients in the linear combination (4.40) are obtained simply as:

$$\alpha_k = \langle \mathbf{x}^{(k)}, \mathbf{y} \rangle \tag{4.42}$$

The coefficients $\{\alpha_k\}$ are called the *Fourier coefficients*[5] of the orthonormal expansion of $\mathbf{y}$ with respect to the basis W and (4.42) is called the Fourier *analysis formula*; conversely, Equation (4.40) is called the *synthesis formula.*

Parseval's Identity For an orthonormal basis, there is a norm conservation property given by *Parseval's identity*:

$$\|\mathbf{y}\|^2 = \sum_{k=0}^{K-1} |\langle \mathbf{x}^{(k)}, \mathbf{y}\rangle|^2. \tag{4.43}$$

For physical quantities, the norm is dimensionally equivalent to a measure of energy; accordingly, Parseval's identity is also known as the *energy conservation formula.*

Basis Completion Let $G = \left\{\mathbf{z}^{(\ell)}\right\}_{\ell=0}^{L-1}$ be a set of orthonormal vectors in a subspace P of dimension $K > L$. Clearly, G is not a basis for P. If $\left\{\mathbf{x}^{(k)}\right\}_{k=0}^{K-1}$ form an orthonormal basis, then we can find vectors $\left\{\tilde{\mathbf{z}}^{(\ell)}\right\}_{\ell=L}^{K-1}$ such that $\left\{\left\{\mathbf{z}^{(\ell)}\right\}_{\ell=0}^{L-1}, \left\{\tilde{\mathbf{z}}^{(\ell)}\right\}_{\ell=L}^{K-1}\right\}$ form an orthonormal basis for P. This can be done in the following manner, since $\left\{\mathbf{x}^{(k)}\right\}_{k=0}^{K-1}$ is an orthonormal basis for P, we want to find $\tilde{\mathbf{z}}^{(L)} \in P$ s.t.

$$\tilde{\mathbf{z}}^{(L)} = \sum_{k=0}^{K-1} \alpha_k \mathbf{x}^{(k)} \quad , \quad \langle \tilde{\mathbf{z}}^{(L)}, \mathbf{z}^{(\ell)}\rangle = 0, \quad \ell = 0, \cdots, L-1$$

$$\Longrightarrow \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{x}^{(k)}, \mathbf{z}^{(\ell)}\rangle = 0, \quad \ell = 0, \cdots, L-1$$

$$\Longrightarrow \begin{bmatrix} \langle \mathbf{x}^{(0)}, \mathbf{z}^{(0)}\rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(0)}\rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}^{(0)}, \mathbf{z}^{(L-1)}\rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(L-1)}\rangle \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix} = \mathbf{0}.$$

[5]Fourier coefficients often refer to the particular case of Fourier series. However, the term generally refers to coefficients in any orthonormal basis.

Since $\begin{bmatrix} \langle \mathbf{x}^{(0)}, \mathbf{z}^{(0)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(0)} \rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}^{(0)}, \mathbf{z}^{(L-1)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(L-1)} \rangle \end{bmatrix} \in \mathbb{C}^{L \times K}$,
we can find a vector in its null space (which is of dimension $K-L$) to obtain $\tilde{\mathbf{z}}^{(L)}$ (properly normalized). In a similar manner we can find $\tilde{\mathbf{z}}^{(\ell)}, \quad \ell = L+1, \cdots, K-1$ by

$$\langle \tilde{\mathbf{z}}^{(\ell)}, \mathbf{z}^{(p)} \rangle = 0, \quad \text{for } p = 0, \cdots, L-1$$

$$\langle \tilde{\mathbf{z}}^{(\ell)}, \tilde{\mathbf{z}}^{(q)} \rangle = 0, \quad \text{for } q = L, \cdots, \ell-1.$$

This amounts to finding a vector in the null space of

$$\begin{bmatrix} \langle \mathbf{x}^{(0)}, \mathbf{z}^{(0)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(0)} \rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}^{(0)}, \mathbf{z}^{(L-1)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \mathbf{z}^{(L-1)} \rangle \\ \langle \mathbf{x}^{(0)}, \tilde{\mathbf{z}}^{(L)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \tilde{\mathbf{z}}^{(L)} \rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}^{(0)}, \tilde{\mathbf{z}}^{(\ell-1)} \rangle & \cdots & \langle \mathbf{x}^{(K-1)}, \tilde{\mathbf{z}}^{(\ell-1)} \rangle \end{bmatrix}$$

Therefore, by *defining* $\mathbf{z}^{(\ell)} = \tilde{\mathbf{z}}^{(\ell)}, \quad \ell = L, \cdots, K-1$, we get an orthonormal basis $\left\{\mathbf{z}^{(\ell)}\right\}_{\ell=0}^{K-1}$ for P, which extends the orthonormal set $\left\{\mathbf{z}^{(\ell)}\right\}_{\ell=0}^{L-1}$ to a basis spanning P. Therefore for any $\mathbf{y} \in P$,

$$\|\mathbf{y}\|^2 = \sum_{k=0}^{K-1} |\langle \mathbf{z}^{(k)}, \mathbf{y} \rangle|^2 \geq \sum_{\ell=0}^{L-1} |\langle \mathbf{z}^{(\ell)}, \mathbf{y} \rangle|^2.$$

Best Approximations. Assume P is a subspace of V; if we try to approximate a vector $\mathbf{y} \in V$ by a linear combination of basis vectors from the subspace P, then we are led to the concepts of least squares approximations and orthogonal projections. First of all, we define the *best* linear approximation $\hat{\mathbf{y}} \in P$ of a general vector $\mathbf{y} \in V$ to be the approximation which minimizes the norm $\|\mathbf{y}-\hat{\mathbf{y}}\|$. Such an approximation is easily obtained by projecting $\mathbf{y}$ onto an orthonormal basis for P, as shown in Figure 4.3. With W, our usual orthonormal basis for P, the projection is given by:

$$\hat{\mathbf{y}} = \sum_{k=0}^{K-1} \langle \mathbf{x}^{(k)}, \mathbf{y} \rangle \mathbf{x}^{(k)}. \tag{4.44}$$

Define the approximation error as the vector $\mathbf{d} = \mathbf{y} - \hat{\mathbf{y}}$. The best approximation $\hat{\mathbf{y}} \in P$ is such that the error $\mathbf{d} = \mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to all vectors $\mathbf{z} \in P$, i.e., $\mathbf{y} - \hat{\mathbf{y}} \perp \mathbf{z} \quad \forall \mathbf{z} \in P$.

Proof: Let $\hat{\mathbf{y}}$ be the unique vector such that $\mathbf{d} = \mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to all vectors $\mathbf{z} \in P$. Consider any vector $\mathbf{g} \in P$ which is a candidate for the best approximation to $\mathbf{y}$ in P.

$$\|\mathbf{y} - \mathbf{g}\|^2 = \|(\mathbf{y} - \hat{\mathbf{y}}) + (\hat{\mathbf{y}} - \mathbf{g})\|^2 = \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}} - \mathbf{g}\|^2 + \langle \mathbf{y} - \hat{\mathbf{y}}, \hat{\mathbf{y}} - \mathbf{g} \rangle + \langle \hat{\mathbf{y}} - \mathbf{g}, \mathbf{y} - \hat{\mathbf{y}} \rangle.$$

However, since $\hat{\mathbf{y}}, \mathbf{g} \in P, \quad \hat{\mathbf{y}} - \mathbf{g} \in P$ and hence

$$\langle \hat{\mathbf{y}} - \mathbf{g}, \mathbf{y} - \hat{\mathbf{y}} \rangle = 0 = \langle \mathbf{y} - \hat{\mathbf{y}}, \hat{\mathbf{y}} - \mathbf{g} \rangle.$$

Hence we have

$$\|\mathbf{y} - \mathbf{g}\|^2 = \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}} - \mathbf{g}\|^2 \geq \|\mathbf{y} - \hat{\mathbf{y}}\|^2$$

with equality iff $\mathbf{g} = \hat{\mathbf{y}}$. $\blacksquare$

Hence, the approximation minimizes the error square norm, i.e.

$$\arg\min_{\hat{\mathbf{y}} \in P} \|\mathbf{y} - \hat{\mathbf{y}}\|^2 = \mathbf{d}. \tag{4.45}$$

This approximation with an orthonormal basis has a key property: it can be *successively refined*. Assume you have the approximation with the first m terms of the orthonormal basis:

$$\hat{\mathbf{y}}_{m-\text{term}} = \sum_{k=0}^{m-1} \langle \mathbf{x}^{(k)}, \mathbf{y} \rangle \mathbf{x}^{(k)} \tag{4.46}$$

and now you want to compute the $(m+1)$-term approximation. This is simply given by

$$\hat{\mathbf{y}}_{(m+1)-\text{term}} = \sum_{k=0}^{m} \langle \mathbf{x}^{(k)}, \mathbf{y} \rangle \mathbf{x}^{(k)} = \hat{\mathbf{y}}_{m-\text{term}} + \langle \mathbf{x}^{(m)}, \mathbf{y} \rangle \mathbf{x}^{(m)}. \tag{4.47}$$

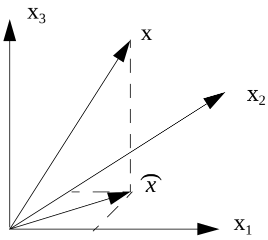


Figure 4.3: Orthogonal projection of the vector $\mathbf{x}$ onto the subspace W spanned by $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$, leading to the approximation $\hat{\mathbf{x}}$. This approximation minimizes the square norm $\|\mathbf{x} - \hat{\mathbf{x}}\|_2$ among all approximations belonging to W.

While this seems obvious, it is actually a small miracle, since it does not hold for more general, non-orthonormal bases.

Bessel's Inequality. The generalization of Parseval's equality is *Bessel's inequality*. Suppose $M = \left\{\mathbf{e}^{(k)}\right\}$ is an orthonormal set, but not a basis for the entire space V. Define a subspace:

$$P = \mathbf{Span}(M) = \mathbf{Span}\left\{\left\{\mathbf{e}^{(k)}\right\}\right\}.$$

Let us consider the projection of an arbitrary vector $\mathbf{x} \in V$, onto P. We have seen that this is given by

$$\hat{\mathbf{x}} = \sum_k \langle \mathbf{e}^{(k)}, \mathbf{x} \rangle \mathbf{e}^{(k)}$$

and that $(\mathbf{x} - \hat{\mathbf{x}}) \perp \mathbf{g}$, $\forall \mathbf{g} \in P$. In particular, since $\hat{\mathbf{x}} \in P$, we see that

$$\mathbf{z} \triangleq \mathbf{x} - \hat{\mathbf{x}} \quad , \quad \mathbf{z} \perp \hat{\mathbf{x}} \quad \text{and}$$

$$\mathbf{x} = \underbrace{\mathbf{x} - \hat{\mathbf{x}}}_{\mathbf{z}} + \hat{\mathbf{x}} = \mathbf{z} + \hat{\mathbf{x}}.$$

Hence by Pythagoras theorem:

$$\|\mathbf{x}\|^2 = \|\mathbf{z}\|^2 + \|\hat{\mathbf{x}}\|^2$$

or: $0 \leq \|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 - \|\hat{\mathbf{x}}\|^2$

or: $\|\hat{\mathbf{x}}\|^2 \overset{(a)}{=} \sum_k |\langle \mathbf{e}^{(k)}, \mathbf{x}\rangle|^2 \leq \|\mathbf{x}\|^2$

where (a) follows from Parseval's relationship.

Hence for any orthonormal set $\{\mathbf{e}^{(k)}\}$,

$$\sum_k |\langle \mathbf{e}^{(k)}, \mathbf{x}\rangle|^2 \leq \|\mathbf{x}\|^2 \qquad \longrightarrow \qquad \text{Bessel's inequality.} \tag{4.48}$$

4.3.3 Examples of Bases

Considering the examples of Section 4.2.2, we have the following:

Finite Euclidean Spaces. For the simplest case of Hilbert spaces, namely $\mathbb{C}^N$, orthonormal bases are also the most intuitive since they contain exactly N mutually orthogonal vectors of unit norm. The classical example is the canonical basis $\{\boldsymbol{\delta}^{(k)}\}_{k=0\ldots N-1}$ with

$$\boldsymbol{\delta}^{(k)}_n = \delta[n-k], \tag{4.49}$$

but we have already seen more interesting bases such as the Fourier basis $\{\mathbf{w}^{(k)}\}$, for which

$$\mathbf{w}^{(k)}_n = e^{j\frac{2\pi}{N}nk}.$$

In $\mathbb{C}^N$, the analysis and synthesis formulas (4.42) and (4.40) take a particularly neat form. For any set $\{\mathbf{x}^{(k)}\}$ of N orthonormal vectors one can indeed arrange the conjugates of the basis vectors[6] as the successive rows of an $N \times N$ square matrix $\mathbf{M}$ so that each matrix element is

$$\mathbf{M}_{mn} = (\mathbf{x}^{(m)}_n)^*;$$

$\mathbf{M}$ is called a *change of basis* matrix. Given a vector $\mathbf{y}$, the set of expansion coefficient $\{\alpha_k\}_{k=0\ldots N-1}$ can now be written *itself* as a vector[7] $\boldsymbol{\alpha} \in \mathbb{C}^N$. Therefore, we can rewrite the analysis formula (4.42) in matrix-vector form and we have:

$$\boldsymbol{\alpha} = \mathbf{M}\mathbf{y}. \tag{4.50}$$

[6]Other definitions may build $\mathbf{M}$ by stacking the *non*-conjugated basis vectors instead; the procedure is however entirely equivalent. Here we choose this definition in order to be consistent with the usual derivation of the Discrete Fourier Transform, which we have seen in Chapter 3.

[7]This isomorphism is rather special and at the foundation of Linear Algebra. If the original vector space V is not $\mathbb{C}^N$, the analysis formula will always provide us with a vector of complex values, but this vector will *not* be in V.

The reconstruction formula (4.40) for $\mathbf{y}$ from the expansion coefficients becomes in turn:

$$\mathbf{y} = \mathbf{M}^H \boldsymbol{\alpha} \tag{4.51}$$

where the superscript denotes the Hermitian transpose (transposition and conjugation of the matrix). The previous equation shows that $\mathbf{y}$ is a linear combination of the columns of $\mathbf{M}^H$, which, in turn, are of course the vectors $\{\mathbf{x}^{(k)}\}$. The orthogonality relation (4.49) takes the following forms

$$\mathbf{M}^H\mathbf{M} = \mathbf{I} \tag{4.52}$$

$$\mathbf{M}\mathbf{M}^H = \mathbf{I} \tag{4.53}$$

since left inverse equals right inverse for square matrices; this implies that $\mathbf{M}$ has orthonormal rows as well as orthonormal columns.

Polynomial Functions. A basis for $\mathbb{P}_N([0,1])$ is $\{x^k\}_{0 \le k < N}$. This basis, however, is not an orthonormal basis. It can be transformed to an orthonormal basis by a standard Gram-Schmidt procedure; the basis vectors thus obtained are called *Legendre polynomials*.

Square Summable Functions. An orthonormal basis set for $L_2([-\pi,\pi])$ is the set $\{(1/\sqrt{2\pi})e^{jnt}\}_{n\in\mathbb{Z}}$. This is actually the classic Fourier basis for functions on an interval. Please note that here, as opposed to the previous examples, the number of basis vectors is actually infinite. The orthogonality of these basis vectors is easily verified; their completeness, however, is extremely hard to prove and this, unfortunately, is pretty much the rule for all non-trivial infinite-dimensional basis sets.

4.4 Signal Spaces Revisited

We are now in the position to formalize our intuitions so far with respect to the equivalence between discrete-time signals and vector spaces, with a particularization for the three main classes of signals we have introduced in Chapter 2. Note that in the following, we will liberally interchange the notations $\mathbf{x}$ and $x[n]$ to denote a sequence as a vector embedded in its appropriate Hilbert space.

4.4.1 Finite-Length Signals

The correspondence between the class of finite-length, length-N signals and $\mathbb{C}^N$ should be so immediate at this point that it does not need further comment. As a reminder, the canonical basis is the canonical basis for $\mathbb{C}^N$. The k-th canonical basis vector will often be expressed in signal form as:

$$\delta[n-k] \quad n = 0,\ldots,N-1,\ k = 0,\ldots,N-1.$$

4.4.2 Periodic Signals

As we have seen, N-periodic signals are equivalent to length-N signals. The space of N-periodic sequences is therefore isomorphic to $\mathbb{C}^N$. In particular, the sum between two sequences considered as vectors is the standard pointwise sum for the elements:

$$z[n] = x[n] + y[n] \quad n \in \mathbb{Z} \tag{4.54}$$

while, for the inner product, we extend the summation over a period only:

$$\langle x[n], y[n] \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]. \tag{4.55}$$

The canonical basis for the space of N-periodic sequences is the canonical basis for $\mathbb{C}^N$, because of the isomorphism; in general, any basis for $\mathbb{C}^N$ is also a basis for the space of N-periodic sequences. Sometimes, however, we will also consider an explicitly *periodized* version of the basis. For the canonical basis, in particular, the periodized basis is composed of N vectors of infinite length $\{\tilde{\boldsymbol{\delta}}^{(k)}\}_{k=0\ldots N-1}$ with:

$$\tilde{\boldsymbol{\delta}}^{(k)} = \sum_{i=-\infty}^{\infty} \delta[n-k-iN].$$

Such a sequence is called a *pulse train*. Note that here we are abandoning mathematical rigor, since the norm of each of these basis vectors is infinite; yet the pulse train, if handled with care, can be a useful tool in formal derivations.

4.4.3 Inifinite Sequences

Finally, this is where we have been heading to all along. For infinite sequences, whose "natural" Euclidean space would appear to be $\mathbb{C}^\infty$, the situation is rather delicate. While the sum of two sequences can be defined in the usual way, by extending the sum for $\mathbb{C}^N$ to $\mathbb{C}^\infty$, care must be taken when evaluating the inner product. We already pointed out that

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n] \tag{4.56}$$

can diverge even for simple constant sequences such as $x[n] = y[n] = 1$. A way out of this impasse is to restrict ourselves to $l_2(\mathbb{Z})$, the space of *square summable sequences*, for which

$$\|x\|^2 = \sum_{n\in\mathbb{Z}} |x[n]|^2 < \infty. \tag{4.57}$$

This is the space of choice for all the theoretical derivations involving infinite sequences. Note that these sequences are often called "of finite energy", since the square norm corresponds to the definition of energy as given in (2.17).

The canonical basis for $l_2(\mathbb{Z})$ is simply the set $\{\boldsymbol{\delta}^{(k)}\}_{k\in\mathbb{Z}}$; in signal form:

$$\boldsymbol{\delta}^{(k)} = \delta[n-k], \quad n, k \in \mathbb{Z}. \tag{4.58}$$

This is an infinite set, and actually an infinite set of linearly independent vectors, since

$$\delta[n-k] = \sum_{l\in\mathbb{Z}/k} \alpha_l \delta[n-l] \tag{4.59}$$

has no solution for any k. Note that, for an arbitrary signal $x[n]$ the analysis formula gives

$$\alpha_k = \langle \boldsymbol{\delta}^{(k)}, \mathbf{x} \rangle = \langle \delta[n-k], x[n] \rangle = x[k]$$

so that the reconstruction formula becomes

$$x[n] = \sum_{k=-\infty}^{\infty} \alpha_k \boldsymbol{\delta}^{(k)} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

which is the reproducing formula (2.16). The Fourier basis for $l_2(\mathbb{Z})$ will be introduced and discussed at length in the next chapter.

As a last remark, note that the space of *all* finite-support signals, which is clearly a subset of $l_2(\mathbb{Z})$, does *not* form a Hilbert space. Clearly, the space is closed under addition and scalar multiplication, and the canonical inner product is well behaved since all sequences have only a finite number of nonzero values. The space however is not complete; to see this, consider the following family of signals

$$y_k[n] = \begin{cases} 1/n & |n| < k \\ 0 & \text{otherwise} \end{cases}$$

For k growing to infinity the sequence of signals converges as $y_k[n] \to y[n] = 1/n$ for all n; while $y[n]$ is indeed in $l_2(\mathbb{Z})$, since

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$y[n]$ is clearly *not* a finite-support signal.

4.5 Summary

The purpose of this chapter was to lay a solid geometrical and algebraic foundation to the theory of discrete-time signal processing. This was achieved by establishing a correspondence between signals and vectors in a Hilbert space. The main points we have covered are:

- A review of Euclidean geometry in the context of simple vector spaces such as $\mathbb{R}^N$.
- The step-by-step definition of Hilbert space as a complete, inner product vector space.
- The properties of inner product and their extension to the concept of norm and distance.
- The concepts of basis and orthonormal basis, with a special emphasis on the latter.
- Examples of Hilbert space: $\mathbb{C}^N$, $P_N([0,1])$, $L_2([-\pi,\pi])$ with their respective inner products and bases.
- A correspondence between the three main classes of discrete-time signals and three suitable types of Hilbert space. In particular, $\mathbb{C}^N$ for finite-length and periodic sequences and $l_2(\mathbb{Z})$ for infinite sequences.

4.6 Problems

Problem 4.1 *Consider the Fourier basis* $\{\boldsymbol{w}^{(k)}\}_{k=0,\ldots,N-1}$*, defined as:*

$$\boldsymbol{w}_n^{(k)} = e^{-j\frac{2\pi}{N}nk}.$$

1. *Prove that it is an orthogonal basis in* $\mathbb{C}^N$.
2. *Normalize the vectors in order to get an orthonormal basis.*

Chapter 5

The DTFT (Discrete-Time Fourier Transform)

We will now consider a Fourier representation for infinite non-periodic sequences. Let us start out abruptly: the Discrete-Time Fourier Transform of a sequence $x[n]$ is defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}. \tag{5.1}$$

The DTFT is therefore a complex-valued function of the *real* argument ω, and, as can be easily verified, it is periodic in ω with period 2π. The somewhat odd notation $X(e^{j\omega})$ is quite standard in the signal processing literature and offers several advantages:

- it stresses the basic periodic nature of the transform, since, obviously, $e^{j(\omega+2\pi)} = e^{j\omega}$;
- regardless of context, it immediately identifies a function as the Fourier transform of a discrete-time sequence: something like $U(e^{j\lambda})$ is just as readily recognizable;
- it provides a nice notational framework which unifies the Fourier transform and the z-transform (which we will see later).

The DTFT, when it exists, can be inverted via the integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{5.2}$$

This can be easily verified by substituting (5.1) into (5.2) and using

$$\int_{-\pi}^{\pi} e^{-j\omega(n-k)} = 2\pi\delta[n-k].$$

In fact, due to the 2π periodicity of the DTFT, the integral in (5.2) can be computed over *any* 2π-wide interval on the real line (i.e. between 0 and 2π, for instance). In general, the relation between a sequence $x[n]$ and its DTFT $X(e^{j\omega})$ will be indicated case by

$$x[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$$

While the DFT and DFS were signal transformation which involved only a finite number of quantities, both the infinite summation and the real-valued argument appearing in the DTFT can create an uneasiness which overshadows the conceptual similarities between the transforms. In the following, we will start by defining the mathematical properties of the DTFT and we will try to build an intuitive feeling for this Fourier representation both with respect to its physical interpretation and to its conformity to the "change of basis" framework we used for the DFT and DFS.

Mathematically, the DTFT is a transform operator which maps discrete-time sequences into the space of 2π-periodic functions. Clearly, for the DTFT to exist, the sum in (5.1) must converge, *i.e.,* the limit for $M \to \infty$ of the partial sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-jwn} \tag{5.3}$$

must exist and be finite. Convergence of the partial sum in (5.3) is very easy to prove for *absolutely summable* sequences, that is for sequences satisfying

$$\lim_{M\to\infty} = \sum_{n=-M}^{M} |x[n]| < \infty \tag{5.4}$$

since

$$|X_M(e^{j\omega})| \le \sum_{n=-M}^{M} |x[n]e^{-jwn}| = \sum_{n=-M}^{M} |x[n]| \tag{5.5}$$

For this class of sequences it can be also proved that the convergence of $X_M(e^{j\omega})$ to $X(e^{j\omega})$ is uniform and that $X(e^{j\omega})$ is continuous. While absolute summability is a sufficient condition, it can be also shown that the sum in (5.3) is convergent for all *square-summable* sequences, *i.e.,* for sequences whose energy is finite; this is very important to us with respect to the discussion in Section 4.4.3, where we defined the Hilbert space $l_2(\mathbb{Z})$. In the case of square summability only, however, the convergence of (5.3) is no longer uniform but takes place only in the mean-square sense, *i.e.,*

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X_M(e^{j\omega}) - X(e^{j\omega})|^2 d\omega = 0. \tag{5.6}$$

Convergence in the mean square sense implies that, while the total energy of the error signal becomes zero, the pointwise values of the partial sum may never approach the values of the limit. One manifestation of this odd behavior is called the *Gibbs phenomenon*, which will have important consequences in our approach to filter design, as we will see later. Furthermore, in the case of square-summable sequences, $X(e^{j\omega})$ is no longer guaranteed to be continuous.

5.1 The DTFT as the Limit of a DFS

A way to gain some intuition about the structure of the DTFT formulas is to consider the DFS of periodic sequences with longer and longer period. Intuitively, as we look at the structure of the Fourier basis for the DFS, we can see that the number of basis vectors in (3.19) grows with the length of the period, N, and, consequently, the frequencies of the underlying complex exponentials become "denser" between 0 and 2π. We want to show that, in the limit, we end up with the reconstruction formula of the DTFT.

To do so, let us restrict ourselves to the domain of absolute summable sequences; for these sequences we know that the sum in (5.1) exists. Now, given an absolutely summable sequence $x[n]$, we can always build an N-periodic sequence $\tilde{x}[n]$ as

$$\tilde{x}[n] = \sum_{i=-\infty}^{\infty} x[n+iN] \tag{5.7}$$

for any value of N; this is guaranteed by the fact that the above sum converges for all $n \in \mathbb{Z}$ (because of the absolute summability of $x[n]$) so that all values of $\tilde{x}[n]$ are finite. Clearly, there is overlap between successive copies of $x[n]$; the intuition, however, is the following: since in the end we will consider very large values for N and since $x[n]$ decays rather fast with n (because of its absolute summability),the resulting overlap of "tails" will be negligible. In other words, we have

$$\lim_{N\to\infty} \tilde{x}[n] = x[n].$$

Consider now the DFS of $\tilde{x}[n]$:

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N-1} \left(\sum_{i=-\infty}^{\infty} x[n+iN] \right) e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{i=-\infty}^{\infty} \left(\sum_{n=0}^{N-1} x[n+iN] e^{-j\frac{2\pi}{N}(n+iN)k} \right) \qquad (5.8)
\end{aligned}$$

where in the second equality we have used (5.7), and the last term is obtained by interchanging the order of the summation and exploiting the fact that $e^{-j(2\pi/N)(n+iN)k} = e^{-j(2\pi/N)nk}$. We can see that, for every value of i in the outer sum, the argument of the inner sum varies between iN and $iN+N-1$, *i.e.,* non-overlapping intervals which cover all the integer numbers, so that the double summation can be simplified as

$$\tilde{X}[k] = \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}mk}, \tag{5.9}$$

and therefore

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=\frac{2\pi}{N}k}. \tag{5.10}$$

This already gives us a noteworthy piece of intuition: the DFS coefficients for the periodized signal are a discrete set of values of its DTFT (here considered solely as a formal operator) computed at multiples of $2\pi/N$. As N grows, the spacing between these frequency intervals narrows more and more so that, in the limit, the DFS converges to the DTFT.

To see that this assertion is consistent, we can now write the DFS reconstruction formula (3.20) and using the DFS values given by (5.10)

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi}{N}k}) e^{j\frac{2\pi}{N}nk}. \tag{5.11}$$

By defining $\Delta = (2\pi/N)$, we can rewrite the above expression as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j(k\Delta)}) e^{j(k\Delta)n} \Delta \tag{5.12}$$

and the summation is easily recognized as the Riemann sum with step Δ approximating the integral of $f(\omega) = X(e^{j\omega})e^{j\omega n}$ between 0 and 2π. As N goes to infinity (and therefore $\tilde{x}[n] \to x[n]$), we can therefore write

$$\tilde{x}[n] \to \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{5.13}$$

which is indeed the DTFT reconstruction formula (5.2)[1].

Example 5.1 *Consider the signal $x[n]$ shown in Fig. 5.1. We can build a periodic signal with period $N = 3$ based on $x[n]$ which is shown in Fig. 5.2.*

[1] Clearly (5.13) is equivalent to (5.2) in spite of the different integration limits since all the quantities under the integral sign are 2π-periodic and we are integrating over a period.

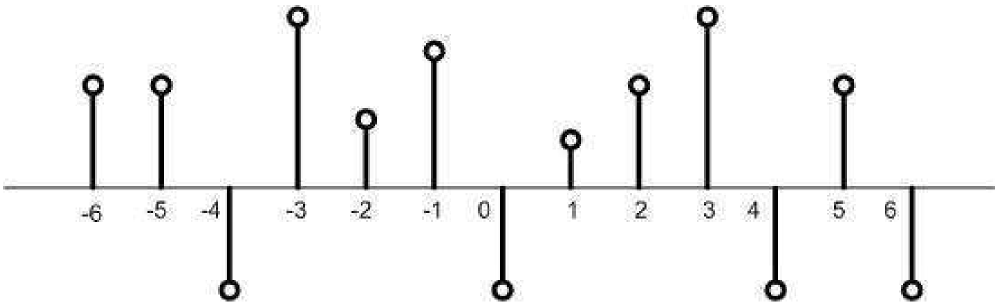


Figure 5.1: $x[n]$

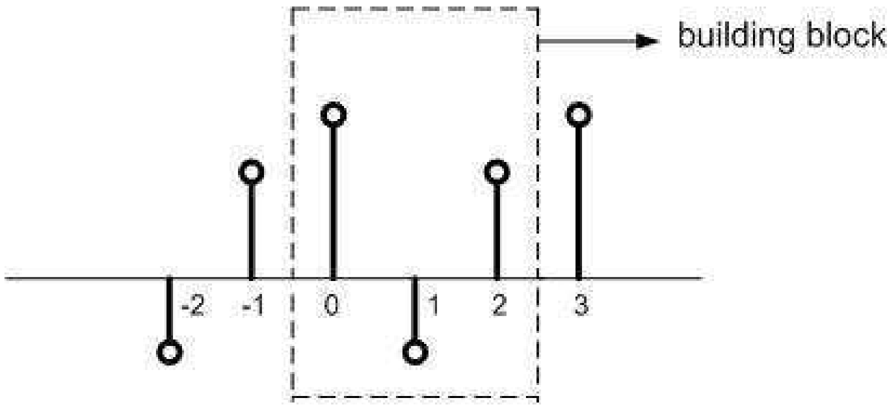


Figure 5.2: $\tilde{x}[n], N = 3$

5.2 The DTFT as a Formal Change of Basis

We will now show that, if we are willing to sacrifice mathematical rigor, the DTFT can be cast in the same conceptual framework we used for the DFT and DFS, namely as a basis change in a vector space. The following formulas are to be taken as nothing more than a set of purely symbolic derivations, since the mathematical hypotheses under which the results are well defined are far from obvious and are completely hidden by the formalism. It is only fair to say, however, that the following expressions represent a very handy and intuitive toolbox to grasp the essence of the duality between the discrete-time and the frequency domains and that they can be put to use very effectively to derive quick results when manipulating sequences.

One way of interpreting equation (5.1) is to see that, for any given value ω_0, the corresponding value of the DTFT is the inner product in $l_2(\mathbb{Z})$ of the sequence $x[n]$ with the sequence $e^{j\omega_0 n}$; at least formally, we are still performing a projection in a vector space akin to $\mathbb{C}^\infty$

$$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle.$$

Here, however, the set of "basis vectors" $\{e^{j\omega n}\}_{\omega\in\mathbb{R}}$ is indexed by the real variable ω and is therefore uncountable. This uncountability is mirrored in the inversion formula (5.2), in which the usual summation is replaced by an integral; in fact, the DTFT operator maps $l_2(\mathbb{Z})$ onto $L_2([-\pi,\pi])$ which is a space of 2π-periodic, square integrable functions. This interpretation preserves the physical meaning given to the inner products in (5.1) as a way to measure the frequency content of the signal at a given frequency; in this case the number of oscillators is infinite and their frequency separation becomes infinitesimally small.

To complete the picture of the DTFT as a change of basis, we want to show that, at least formally, the set $\{e^{j\omega n}\}_{\omega\in\mathbb{R}}$ constitutes an orthogonal "basis" for $l_2(\mathbb{Z})$[2]. In order to do so we need to introduce a quirky mathematical entity called the Dirac delta functional; this is defined in an implicit way by the following formula

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t)dt = f(\tau) \tag{5.14}$$

where $f(t)$ is an arbitrary integrable function on the real line; in particular:

$$\int_{-\infty}^{\infty} \delta(t) f(t)dt = f(0). \tag{5.15}$$

While no ordinary function satisfies the above equation, $\delta(t)$ can be interpreted as a shorthand for a limiting operation. Consider for instance the family of parametric functions[3]

$$r_k(t) = k\,\mathrm{rect}(kt) \tag{5.16}$$

which are plotted in Figure 5.3. For any continuous function $f(t)$ we can write

$$\int_{-\infty}^{\infty} r_k(t) f(t)dt = k\int_{-1/2k}^{1/2k} f(t)dt = f(\gamma)|_{\gamma\in[-1/2k,1/2k]} \tag{5.17}$$

[2]You can see here already why this line of thought is shaky: indeed, $e^{j\omega n} \notin l_2(\mathbb{Z})$!

[3]The rect function is introduced in its full glory in Section 7.7.1; it is defined as

$$\mathrm{rect}(x) = \begin{cases} 1 & \text{for } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

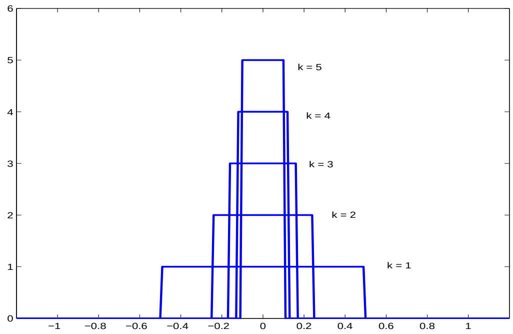


Figure 5.3: The Dirac delta as the limit of a family of rect functions.

where we have used the mean value theorem. Now, as k goes to infinity we have that the integral converges to $f(0)$ and so we can say that the limit of the series of functions $r_k(t)$ converges then to the Dirac delta. The delta, as we said, cannot be considered as a proper function so the expression $\delta(t)$ outside of an integral sign has no mathematical meaning; it is customary however to associate an "idea" of function to the delta and we can think of it as being undefined for $t \neq 0$ and to have a value of ∞ at $t = 0$. This interpretation, together with (5.14), defines the so-called *shifting property* of the Dirac delta; this property allows us to write (outside of the integral sign)

$$\delta(t-\tau)f(t) = f(\tau)\delta(t-\tau) \tag{5.18}$$

The physical interpretation of the Dirac delta is related to quantities expressed as continuous *distributions* for which the most familiar example is probably that of a probability distribution (pdf). These functions represents a value which makes physical sense only over an interval of nonzero measure; the punctual value of a distribution is only an abstraction. The Dirac delta is the operator that extracts this punctual value from a distribution, in a sense capturing the essence of considering smaller and smaller observation intervals.

To see how the Dirac delta applies to our basis expansion, note that equation (5.14) is formally identical to an inner product over the space of functions on the real line; by using the definition of such an inner product we can therefore write

$$f(t) = \int_{-\infty}^{\infty} \langle \delta(s-\tau), f(s) \rangle \, \delta(t-\tau) d\tau \tag{5.19}$$

which is in turn formally identical to the reconstruction formula of Section 4.4.3. In reality, since DTFT's live in the space of 2π-periodic functions, we are interested in this space.

This is easily accomplished by building a 2π-periodic version of the delta as

$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k). \tag{5.20}$$

where the leading 2π factor is for later convenience. The resulting object is called a *pulse train*, similarly to what we built for the case of periodic sequences in $\tilde{\mathbb{C}}^N$. Using the pulse train and given any 2π-periodic function $f(\omega)$, the reconstruction formula (5.19) becomes:

$$f(\omega) = \frac{1}{2\pi} \int_{\sigma}^{\sigma+2\pi} \langle \tilde{\delta}(\theta - \phi), f(\theta) \rangle \, \tilde{\delta}(\omega - \phi) d\phi \tag{5.21}$$

for any $\sigma \in \mathbb{R}$.

Now that we have the delta notation in place, we are ready to start. First of all, we will show the formal orthogonality of the basis functions $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$. We can write

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\delta}(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n}. \tag{5.22}$$

The left-hand side of this equation has the exact form of the DTFT reconstruction formula (5.2), and so we have found the the fundamental relationship

$$e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} \tilde{\delta}(\omega - \omega_0). \tag{5.23}$$

Recall our change of basis interpretation in which the DTFT of a complex exponential $e^{j\sigma n}$ is simply the inner product $\langle e^{j\omega n}, e^{j\sigma n} \rangle$. Therefore, (5.23) implies

$$\langle e^{j\omega n}, e^{j\sigma n} \rangle = \tilde{\delta}(\omega - \sigma). \tag{5.24}$$

We will now recall for the last time that the delta notation subsumes a limiting operation; the DTFT pair (5.23) should be interpreted as a shorthand for the limit of the partial sums

$$s_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$$

(where we have chosen $\omega_0 = 0$ for the sake of example). Figure 5.4 plots $|s_k(\omega)|$ for increasing values of k (we show only the $[-\pi, \pi]$ interval, although of course the functions are 2π-periodic). The family of functions $s_k(\omega)$ is exactly equivalent to the family of $r_k(t)$'s we saw in (5.16) in the sense that they also become narrower and narrower while keeping a constant area (which turns out to be 2π). That is why we can say simply that $s_k(\omega) \to \tilde{\delta}(\omega)$.

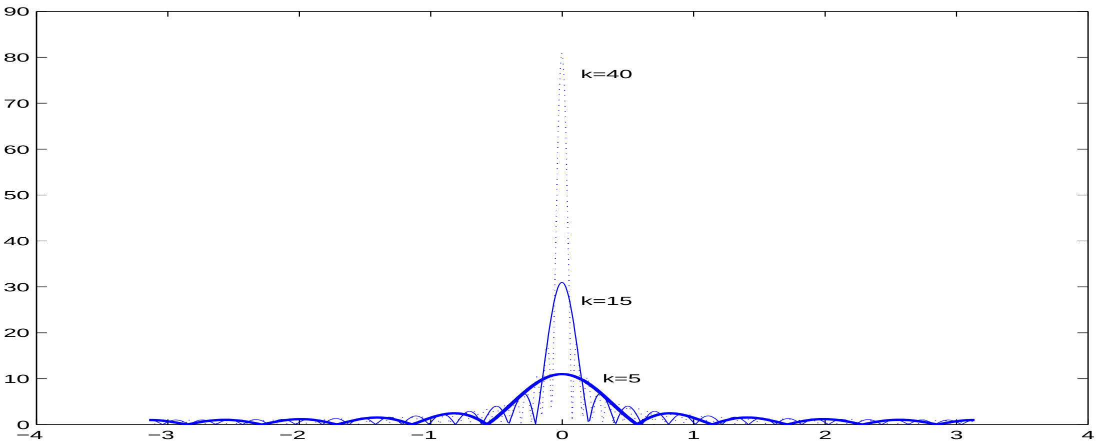


Figure 5.4: Plot of the function $|\sum_{n=-k}^{k} e^{-j\omega n}|$ for different values of k.

From (5.23) we can easily obtain other interesting results; by setting $\omega_0 = 0$ and by exploiting the linearity of the DTFT operator we can derive the DTFT of a constant sequence as

$$\alpha \stackrel{\text{DTFT}}{\longleftrightarrow} \alpha\tilde{\delta}(\omega); \tag{5.25}$$

for any arbitrary $\alpha \in \mathbb{R}$. The DTFT's of sinusoidal functions are also derived using Euler's formulas,

$$\cos(\omega_0 n + \phi) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}\left[e^{j\phi}\tilde{\delta}(\omega - \omega_0) + e^{-j\phi}\tilde{\delta}(\omega + \omega_0)\right], \tag{5.26}$$

$$\sin(\omega_0 n + \phi) \stackrel{\text{DTFT}}{\longleftrightarrow} -\frac{j}{2}\left[e^{j\phi}\tilde{\delta}(\omega - \omega_0) - e^{-j\phi}\tilde{\delta}(\omega + \omega_0)\right]. \tag{5.27}$$

As we can see from the above examples, we are defining the DTFT for sequences which are not even square summable; again, these transforms are purely a notational formalism used to capture a behavior in the limit as we showed before.

5.3 Relationships Between Transforms

We will conclude this section on the DTFT by showing that the DTFT is the most general type of Fourier transform for discrete-time signals. Consider a length-N signal $x[n]$ and its N-DFT coefficients $X[k]$; consider also the sequences obtained from $x[n]$ either by periodization or by building a finite-support sequence. The computation of the DTFT's of these sequences will highlight the relationships linking the three types of discrete-time transforms we have seen so far.

Periodic Sequences. Given a length-N signal $x[n]$, $n = 0, \ldots, N-1$, consider the associated N-periodic sequence $\tilde{x}[n] = x[n \mod N]$ and its N DFS coefficients $X[k]$. If we write the analysis DTFT formula for $\tilde{x}[n]$ we have

$$\begin{aligned}
\tilde{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \tilde{x}[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} && (5.28) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right), && (5.29)
\end{aligned}$$

where in (5.28) we have used the DFS reconstruction formula. Now it is immediate to recognize in the last term of (5.29) as the DTFT of a complex exponential of frequency $(2\pi/N)k$; we can therefore write

$$\tilde{X}(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\tilde{\delta}(\omega - \frac{2\pi}{N}k), \tag{5.30}$$

which gives the relationship between the DTFT and the DFS. If we restrict ourselves to the $[-\pi, \pi]$ interval, we can see that the DTFT of a periodic sequence is a series of regularly spaced deltas placed at the N roots of unity and whose amplitude is proportional to the DFS coefficients of the sequence. In other words, *the DTFT of a priodic sequence is uniquely determined by its DFS and vice versa.*

Finite-Support Sequences. Given a length-N signal $x[n], n = 0, \ldots, N-1$ and its N DFT coefficients $X[k]$, consider the associated finite-support sequence

$$\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N, \\ 0 & \text{otherwise.} \end{cases} ;$$

We can easily derive the DTFT of $\bar{x}$ as

$$\begin{aligned}
\bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n} \\
&= \sum_{k=0}^{N-1} X[k] \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right) \\
&= \sum_{k=0}^{N-1} X[k]\Lambda(\omega - \frac{2\pi}{N}k) \qquad (5.31)
\end{aligned}$$

with

$$\Lambda(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n}.$$

What the above expression means is that the DTFT of the finite support sequence $\bar{x}[n]$ is again *uniquely determined by the N-DFT coefficients of the finite-length signal* $x[n]$ and it can be obtained by a simple Lagrangian interpolation. As in the previous case, the values of DTFT at the roots of unity are equal to the DFT coefficients; note, however, that the transform of a finite support sequence is very different from the DTFT of a periodized sequence. The latter, in accordance with the definition of the Dirac delta, is defined only in the limit and for a finite set of frequencies; the former is just a (smooth) interpolation of the DFT.

5.4 Properties of the DTFT

The DTFT possesses the following properties.

Symmetries & Structure. The DTFT of a time-reversed sequence is

$$x[-n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(e^{-j\omega}) \qquad (5.32)$$

while, for the complex conjugate of a sequence we have

$$x^*[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X^*(e^{-j\omega}). \qquad (5.33)$$

For the very important case of a *real* sequence $x[n] \in \mathbb{R}$ we have that the DTFT is conjugate-symmetric:

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \tag{5.34}$$

which leads to the following symmetries (again, only for **real** signals)

$$|X(e^{j\omega})| = |X(e^{-j\omega})| \qquad \textit{the magnitude is symmetric} \tag{5.35}$$
$$\measuredangle X(e^{j\omega}) = -\measuredangle X(e^{-j\omega}) \qquad \textit{the phase is antisymmetric} \tag{5.36}$$
$$\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\} \qquad \textit{the real part is symmetric} \tag{5.37}$$
$$\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\} \qquad \textit{the imaginary is antisymmetric} \tag{5.38}$$

Finally, if $x[n]$ is real and symmetric, then the DTFT is real.

$$x[n] \in \mathbb{R},\ x[-n] = x[n] \Longleftrightarrow X(e^{j\omega}) \in \mathbb{R}, \tag{5.39}$$

while, for real antisymmetric signals we have that the DTFT is purely imaginary.

$$x[n] \in \mathbb{R},\ x[-n] = -x[n] \Longleftrightarrow \mathrm{Re}\{X(e^{j\omega})\} = 0. \tag{5.40}$$

Linearity & Shifts. The DTFT is a linear operator:

$$\alpha x[n] + \beta y[n] \overset{\mathrm{DTFT}}{\longleftrightarrow} \alpha X(e^{j\omega}) + \beta Y(e^{j\omega}). \tag{5.41}$$

A shift in the discrete-time domain leads to multiplication by a phase term in the frequency domain:

$$x[n - n_0] \overset{\mathrm{DTFT}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}), \tag{5.42}$$

while multiplication of the signal by a complex exponential (i.e. signal *modulation* by a complex "carrier" at frequency ω_0) leads to

$$e^{j\omega_0 n} x[n] \overset{\mathrm{DTFT}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}), \tag{5.43}$$

which means that the spectrum is shifted by ω_0. This last result is known as the *Modulation Theorem.*

Modulation Property.

$$x[n]y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) \mathrm{d}\theta$$

Proof:

$$\sum_n y[n]x[n]e^{-j\omega n} = \sum_n y[n]\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) e^{j\theta n}\mathrm{d}\theta\, e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)\sum_n y[n]e^{-j(\omega-\theta)n}\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$

Planchard-Parseval equality.

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^*[n]y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)\mathrm{d}\omega \\ \text{or} \qquad \langle \mathbf{x}, \mathbf{y}\rangle &= \frac{1}{2\pi}\langle X, Y\rangle \end{aligned}$$

Proof: From the modulation property, we have

$$z[n] = x^*[n]y[n] \overset{DTFT}{\longleftrightarrow} Z\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{-j\theta}\right) Y\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$

$$\Longrightarrow \sum_n x^*[n]y[n] = \sum_n z[n] = Z\left(e^{j\omega}\right)\Big|_{\omega=0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{-j\theta}\right) Y\left(e^{-j\theta}\right)\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{j\theta}\right) Y\left(e^{j\theta}\right)\mathrm{d}\theta$$

Energy Conservation. The DTFT satisfyes following equality

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \tag{5.44}$$

which establishes the conservation of energy property between the time and the frequency domains. The above equality is known as the *Parseval's Theorem*.

Example 5.2

a) Determine the DTFT of the sequence

$$x[n] = \begin{cases} A & 0 \leq n \leq L-1, \\ 0 & \textit{otherwise}. \end{cases}$$

b) Give an approximate plot of the magnitude and phase of the spectrum (feel free to use MATLAB; create a sequence $x[n]$ of length $N = 1000$ for example, use `fft` to compute the N-point DFT and set $A = 1$ and $L = 5$, check out the commands `abs` and `angle`).

Solution:

a) First, we calculate the DTFT of $x[n]$ as follows:

$$X(e^{j\omega}) = \sum_{n=0}^{L-1} Ae^{-j\omega n} = A\frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = Ae^{-j(\frac{\omega}{2})(L-1)}\frac{\sin(\omega L/2)}{\sin(\omega/2)}.$$

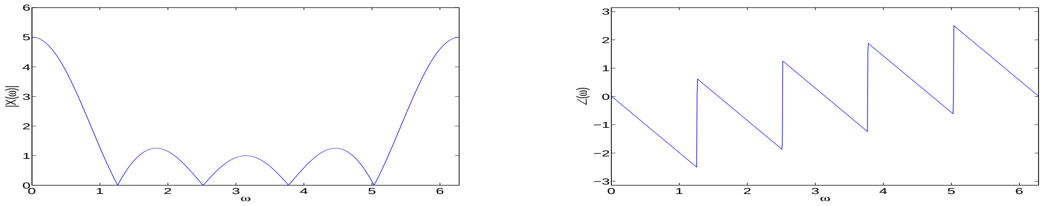
b) The amplitude and phase of $X(\omega)$ are given by:

$$\begin{aligned} |X(e^{j\omega})| &= \begin{cases} |A|L & \textit{if } \omega = 0, \\ |A||\frac{\sin(\omega L/2)}{\sin(\omega/2)}| & \textit{otherwise}. \end{cases} \\ \angle X(e^{j\omega}) &= \angle A - (\omega/2)(L-1) + \angle\frac{\sin(\omega L/2)}{\sin(\omega/2)}, \end{aligned}$$

where we should remember that the phase of a real quantity is zero if the quantity is positive and π if it is negative. The amplitude and phase are plotted in Figure 5.5.

5.5 Summary

This chapter introduced the concept of the discrete-time Fourier Transform. Here is a table of common transforms.



(a) AMPLITUDE (b) PHASE

Figure 5.5: Amplitude and phase of $X(e^{j\omega})$

Some DTFT pairs:

$x[n] = \delta[n-k]$	$X(e^{j\omega}) = e^{-j\omega k}$
$x[n] = 1$	$X(e^{j\omega}) = \tilde{\delta}(\omega)$
$x[n] = u[n]$	$X(e^{j\omega}) = \dfrac{1}{1-e^{-j\omega}} + \dfrac{1}{2}\tilde{\delta}(\omega)$
$x[n] = a^n u[n] \quad \lvert a\rvert < 1$	$X(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$
$x[n] = e^{j\omega_0 n}$	$X(e^{j\omega}) = \tilde{\delta}(\omega-\omega_0)$
$x[n] = \cos(\omega_0 n + \phi)$	$X(e^{j\omega}) = (1/2)[e^{j\phi}\tilde{\delta}(\omega-\omega_0) + e^{-j\phi}\tilde{\delta}(\omega+\omega_0)]$
$x[n] = \sin(\omega_0 n + \phi)$	$X(e^{j\omega}) = (-j/2)[e^{j\phi}\tilde{\delta}(\omega-\omega_0) - e^{-j\phi}\tilde{\delta}(\omega+\omega_0)]$
$x[n] = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$X(e^{j\omega}) = \dfrac{\sin((N/2)\omega)}{\sin(\omega/2)}\, e^{-j\frac{N-1}{2}\omega}$

5.6 Problems

Problem 5.1 *Let $x[n]$ and $y[n]$ be two complex valued sequences and $X(e^{jw})$ and $Y(e^{jw})$ be their corresponding DTFTs.*

(a) Show that

$$\langle x[n], y[n]\rangle = \frac{1}{2\pi}\langle X(e^{jw}), Y(e^{jw})\rangle,$$

where we use the inner products for $l_2(\mathbb{Z})$ and $L_2([-\pi,\pi])$ respectively.

(b) What is the physical meaning of the above formula when $x[n] = y[n]$?

Problem 5.2 (DFT and DTFT) *Consider the infinite non-periodic sequence*

$$x[n] = \begin{cases} 0 & n < 3 \\ 1 & 3 \leq n < 10 \\ -1 & 10 \leq n < 15 \\ 0 & n \geq 15. \end{cases}$$

(a) Derive the DTFT $X(e^{jw})$ of $x[n]$. (Don't use MATLAB.)

(b) Use MATLAB to plot the magnitude of $X(e^{jw})$ for 1000 points in the interval $[0, 2\pi]$.

(c) In MATLAB, use `fft` function with $N = 20$ to compute the DFT of $x[n]$. Compare this plot to one obtained in (b).

(d) Repeat part (c) for $N = 50, 100, 1000$. What can you conclude?

(e) Can you prove your your conclusion analytically?

Problem 5.3 *Let $x[n]$ be a discrete-time sequence defined as*

$$x[n] = \begin{cases} M - n & 0 \leq n \leq M, \\ M + n & -M \leq n \leq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

for some odd integer M.

(a) Show that $x[n]$ can be express as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$. Check your result with MATLAB for $M = 11$.

(b) Using the results found in (a), compute the DTFT of $x[n]$.

Problem 5.4 *Consider the system $\mathcal{H}$ implementing the input-output relation $y[n] = \mathcal{H}\{x[n]\} = x^2[n]$.*

(a) Prove by example that the system is nonlinear.

(b) Prove that the system is time-invariant

Now consider the following cascade system:

$$x[n] \longrightarrow \boxed{\mathcal{H}} \xrightarrow{y[n]} \boxed{\mathcal{G}} \longrightarrow v[n]$$

where $\mathcal{G}$ is the following ideal highpass filter:

$$G(e^{j\omega}) = \begin{cases} 0 & \text{for } |\omega| < \pi/2 \\ 2 & \text{otherwise} \end{cases}$$

(as per usual, $G(e^{j\omega})$ is 2π-periodic (i.e., prolonged by periodicity outside of $[-\pi, \pi]$)). The output of the cascade is therefore $v[n] = \mathcal{G}\{\mathcal{H}\{x[n]\}\}$.

(c) Compute $v[n]$ when $x[n] = \cos(\omega_0 n)$ for $\omega_0 = 3\pi/8$. How would you describe the transformation operated by the cascade on the input?

(d) Compute $v[n]$ as before, with now $\omega_0 = 7\pi/8$