Chapter 10

Interpolation and Sampling

In the introduction to these notes we remarked that discrete-time signals are the mathematical model of choice in two signal processing situations: the first, which encompasses the long-established tradition of observing physical phenomena, captures the process of repeatedly measuring the value of a physical quantity at successive instants in time for *analysis* purposes. The second, which is much more recent and dates back to the first digital processors, is the ability to *synthesize* discrete-time signal by means of iterative numerical algorithms.

The repeated measurement of a "natural" signal is called *sampling*; at the base of the notion is a view of the world in which physical phenomena have a potentially infinitely small granularity, in the sense that measurements can be achieved with arbitrary denseness. For this reason, it is customary to model real-world phenomena as *functions of a real variable* (the variable being time or space); defining a quantity over the real line allows for infinitely small subdivisions of the function's domain and therefore infinitely precise localization of its values. We will refer to this model of the world as to the *continuous-time* paradigm. Whether philosophically valid[1] or physically valid[2], the continuous-time paradigm is a model of immense usefulness in the description of analog signal processing systems. So useful, in fact, that even in the completely discrete-time synthesis scenario, we will often find ourselves in the need of converting a sequence to a well defined function of a continuous variable in order to interface our digital world to the analog world outside. The process, which can be seen as the dual of sampling, is called *interpolation*.

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[1] Remember Zeno's paradoxes...

[2] The shortest unit of time at which the usual laws of gravitational physics still hold is called *Planck time* and is estimated at 10^{-43} seconds. Apparently, therefore, the universe works in discrete-time...

10.1 Preliminaries and Notation

Interpolation. Interpolation comes into play when discrete-time signals need to be converted to continuous-time signals. The need arises at the interface between the digital world and the analog world; as an example, consider a discrete-time waveform synthesizer which is used to drive an analog amplifier and loudspeaker. In this case, it is useful to express the input to the amplifier as a function of a real variable, defined over the entire real line; this is because the behavior of analog circuitry is best modeled by continuous-time functions. We will see that at the core of the interpolation process is the association of a physical time duration T_s to the intervals between samples of the discrete-time sequence. The fundamental questions concerning interpolation involve the spectral properties of the interpolated function with respect to those of the original sequence.

Sampling.

A typical method to obtain a discrete-time representation of a continuous-time signal is through periodic sampling (*uniform sampling*) where a sequence of samples $x[n]$ are obtained from a continuous-time signal $x_c(t)$ as,

$$x[n] = x_c(nT_s), \qquad -\infty < n < \infty \tag{10.1}$$

where T_s is the sampling period and $F_s = \frac{1}{T_s}$ is the sampling frequency.

A natural question we asked is whether such a sampling process extends a loss of information, i.e. given $\{x[n]\}$, can we reconstruct $x_c(t)$ for any t?

This would mean that we can *interpolate* between values of $\{x_c(nT_s)\}$ to reconstruct $x_c(t)$. If the answer is in the negative (at least for a given class of signals), this means that all the processing tools developed in the discrete-time domain can be applied to continuous-time signals as well, after sampling. The fundamental question is whether this is possible, and if so what are the interpolating functions.

Notation. In the rest of this chapter we will encounter a series of variables which are all interrelated and whose different forms will be used interchangeably according to convenience. They are summarized here for a quick reference:

Name	Description	Units	Relations
T_s	Sampling period	seconds	$T_s = 1/F_s$
F_s	Sampling frequency	Hertz	$F_s = 1/T_s$
Ω_s	Sampling frequency (angular)	rad/sec	$\Omega_s = 2\pi F_s = 2\pi/T_s$
Ω_N	Nyquist frequency (angular)	rad/sec	$\Omega_N = \Omega_s/2 = \pi/T_s$

10.2 Continuous-Time signals

Interpolation and sampling constitute the bridges between the discrete- and continuous-time worlds. Before we proceed to the core of the matter, it is useful to take a quick tour of the main properties of continuous-time signals, which we will simply state without formal proofs.

Continuous-time signals are modeled by complex functions of a real variable t which usually represents time (in seconds) but which can represent other physical coordinates of interest. For maximum generality, no special requirement is imposed on functions modeling signals; just as in the discrete-time case, the functions can be periodic or aperiodic, or they can have a finite support (in the sense that they are nonzero over a finite interval only). A common condition on an aperiodic signal is that its modeling function be square integrable; this corresponds to the reasonable requirement that the signal have finite energy.

Inner product and convolution. We have already encountered some examples of continuous-time signals in conjunction with Hilbert spaces; in section 4.2.2, for instance, we introduced the space of square integrable functions over an interval and, in a short while, we will introduce the space of bandlimited signals. For inner product spaces whose elements are functions on the real line, we will use the following inner product definition:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^*(t) g(t) dt \tag{10.2}$$

The *convolution* of two real continuous-time signals is defined as usual from the definition of the inner product; in particular

$$(f * g)(t) = \langle f(t-\tau), g(\tau) \rangle \tag{10.3}$$

$$= \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \tag{10.4}$$

The convolution operator in continuous time is linear and time invariant, as can be easily verified. Note that, just like in discrete-time, convolution represents the operation of filtering a signal with a continuous-time LTI filter, whose impulse response is of course a continuous-time function.

Frequency-Domain Representation of Continuous-Time Signals. The Fourier transform of a continuous-time signal $x(t)$ and its inversion formula are defined as[3]:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt, \tag{10.5}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega. \tag{10.6}$$

The convergence of the above integrals is assured for functions which satisfy the so-called Dirichlet conditions. In particular, the Fourier transform is always well defined for square integrable (finite energy) continuous-time signals. The Fourier transform in continuous time is a linear operator; for a list of its properties, which mirror those we saw for the DTFT, we refer to the bibliography. Suffice here to recall the conservation of energy, also known as Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega.$$

The FT representation can be formally extended to signals which are not square summable by means of the Dirac delta notation as we saw in Section 5.2. In particular we have

$$\text{CTFT}\{e^{j\Omega_0 t}\} = 2\pi\delta(\Omega - \Omega_0), \tag{10.7}$$

from which the Fourier transforms of sine, cosine, and constant functions can be easily derived. Please note that, in continuous-time, the CTFT of a complex exponential is *not* a train of impulses but just a single impulse.

The Convolution Theorem. The convolution theorem for continuous-time signal exactly mirrors the theorem in section 7.5.2; it states that if $h(t) = (f * g)(t)$ then the Fourier transforms of the three signals are related by $H(j\Omega) = F(j\Omega)G(j\Omega)$. In particular we can use the convolution theorem to compute

$$(f * g)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\Omega)G(j\Omega)e^{j\Omega t}d\Omega \tag{10.8}$$

10.3 Bandlimited Signals

A signal whose Fourier transform is nonzero only over a finite (bounded) frequency interval is called *bandlimited*. In other words, the signal $x(t)$ is bandlimited if there exists a

[3]The notation $X(j\Omega)$ mirrors the specialized notation we used for the DTFT; in this case, by writing $X(j\Omega)$ we indicate that the Fourier transform is just the (two-sided) Laplace transform $X(s) = \int x(t)e^{-st}dt$ computed on the imaginary axis.

frequency Ω_N such that[4]

$$X(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N.$$

Such a signal will be called Ω_N-bandlimited and Ω_N is often called the *Nyquist frequency.* It may be useful to mention that, symmetrically, a continuous-time signal which is nonzero over a finite time interval only is called a *time-limited* signal (or finite-support signal). A fundamental theorem states that a bandlimited signal cannot be time-limited, and vice versa. While this can be proved formally without too much effort, here we simply give the intuition behind the statement. The time-scaling property of the Fourier transform states that

$$\text{CTFT}\{f(at)\} = \frac{1}{|a|}\, F(j\frac{\Omega}{a})$$

so that the more "compact" in time a signal is, the wider it frequency support becomes.

The Sinc Function. Let us now consider a prototypical Ω_N-bandlimited signal $\varphi(t)$ whose Fourier transform is *constant* over the interval $[-\Omega_N, \Omega_N]$ and zero everywhere else. If we define the rect function as (see also section 7.7.1):

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

we can express the Fourier transform of the prototypical Ω_N-bandlimited signal as

$$\Phi(j\Omega) = \frac{\pi}{\Omega_N}\text{rect}\left(\frac{\Omega}{2\Omega_N}\right) \tag{10.9}$$

where the leading factor is just a normalization term. The time-domain expression for the signal is easily obtained from the inverse Fourier transform as

$$\varphi(t) = \frac{\sin \Omega_N t}{\Omega_N t} = \text{sinc}\left(\frac{t}{T_s}\right) \tag{10.10}$$

where we have used $T_s = \pi/\Omega_N$ and defined the sinc function as

$$\text{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

The sinc function is plotted in Figure 10.1.

Note the following:

[4]The use of $\geq$ instead of $>$ is a technicality which will be useful in conjunction with the sampling theorem below.

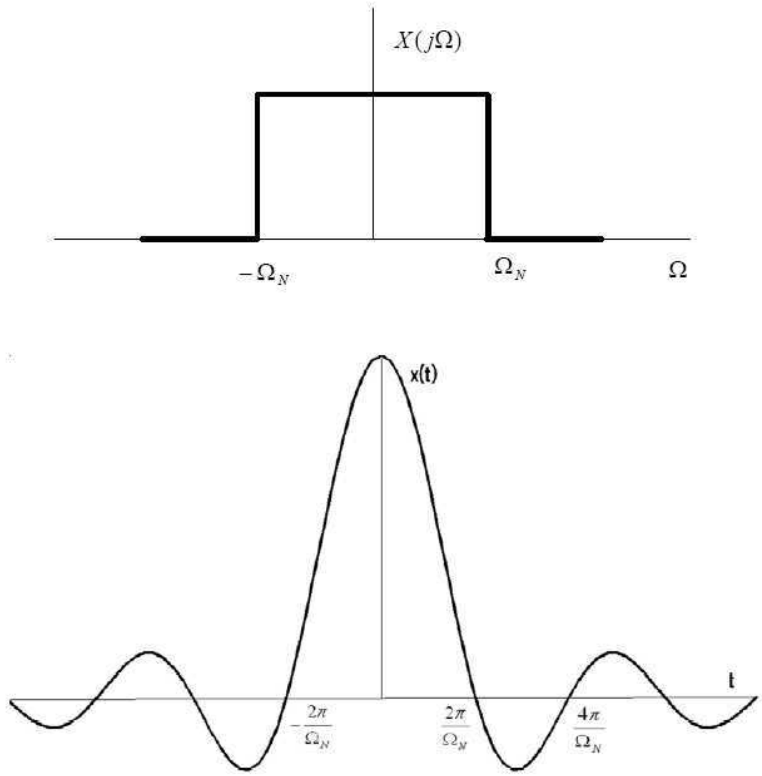


Figure 10.1: The sinc function in frequency ($X(j\Omega)$) and time ($x(t)$) domains.

- The function is symmetric, $\text{sinc}(x) = \text{sinc}(-x)$
- The sinc function is zero for all integer values of its argument, except in zero. This feature is called the *interpolation property* of the sinc, as we will see more in detail later.
- The sinc function is square integrable (it has finite energy) but it is not absolutely integrable (hence the discontinuity of its Fourier transform).
- The decay is slow, asymptotic to $1/x$.
- The scaled sinc function represents the impulse response of an ideal, continuous-time lowpass filter with cutoff frequency Ω_N.

10.4 The Sampling Theorem

We have seen in the previous section that the "natural" polynomial interpolation scheme leads to the so-called sinc interpolation for infinite discrete time sequences. Another way to look at the previous result is that any square summable discrete-time signal can be interpolated into a continuous-time signal which is smooth in time and strictly bandlimited in frequency. This suggests that the class of bandlimited functions must play a special role in bridging the gap between discrete and continuous time and this deserves further investigation. In particular, since any discrete-time signal can be interpolated exactly into a bandlimited function, we now ask ourselves whether the converse is true: can any bandlimited signal be transformed into a discrete-time signal with no loss of information?

10.4.1 Frequency-Domain Representation of Sampling

Given a continuous-time signal $x_c(t)$, we do periodic sampling by producing

$$x[n] = x_c(nT_s) = x_c(t)\big|_{t=nT_s}. \tag{10.11}$$

Let us define a new continuous-time signal which places Dirac delta impulses at the sampling locations, i.e.,

$$x_s(t) = \sum_n x[n]\delta(t - nT_s) = \sum_n x_c(nT_s)\delta(t - nT_s), \tag{10.12}$$

which is a fictitious signal serving as an intermediate step between the continuous and discrete-time worlds.

We can also write

$$x_s(t) = \sum_n x_c(nT_s)\delta(t - nT_s) = x_c(t) \underbrace{\sum_n \delta(t - nT_s)}_{s(t)}, \tag{10.13}$$

i.e.,

$$x_s(t) = x_c(t)s(t). \tag{10.14}$$

Hence we see that from the modulation property of continuous-time Fourier transforms,

$$X_s(j\Omega) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_c(j\theta)S\left(j(\Omega-\theta)\right)\,\mathrm{d}\theta}_{X_c(j\Omega)*S(j\Omega)}. \tag{10.15}$$

Now

$$\sum_n \delta(t - nT_s) \overset{\text{CTFT}}{\Longleftrightarrow} \frac{2\pi}{T_s} \sum_{k=-\infty}^{+\infty} \delta(\Omega - k\Omega_s), \tag{10.16}$$

where $\Omega_s = \frac{2\pi}{T_s}$. Using this in (10.15) we see that

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c\left(j(\Omega - k\Omega_s)\right). \tag{10.17}$$

Therefore, we see that the sampled sequence has a Fourier transform which consists of periodically repeated copies of the original CTFT of $x_c(t)$, shifted by integer multiples and superimposed.

To observe its effects, see Figure 10.2 representing a bandlimited Fourier transform with bandwidth Ω_N, Figure 10.3 is the periodic impulse train $S(j\Omega)$ and finally Figure 10.4 is $X_s(j\Omega)$ along with $X\left(e^{j\omega}\right)$ in Figure 10.5. From Figure 10.4 it is clear that to retain information through sampling we need

$$\Omega_s - \Omega_N > \Omega_N \qquad \text{or} \qquad \Omega_s > 2\Omega_N, \tag{10.18}$$

so that the replicas of $X_c(j\Omega)$ do not overlap when they are added together in (10.17). If this condition is satisfied, it is clear that one can recover $x_c(t)$ from $x[n]$ (or $X_c(j\Omega)$ from $X\left(e^{j\omega}\right)$) by taking the inverse CTFT of one of the replicas, i.e., by taking

$$X_r(j\Omega) = H_r(j\Omega) X_s(j\Omega), \tag{10.19}$$

where

$$H_r(j\Omega) = \begin{cases} T_s & |\Omega| \le \Omega_c \\ 0 & \text{else,} \end{cases} \tag{10.20}$$

and

$$\Omega_N \le \Omega_c \le \Omega_s - \Omega_N.$$

This leads us to the sampling theorem:

If $x_c(t)$ is a bandlimited signal with $X_c(j\Omega) = 0$ for $|\Omega| > \Omega_N$, then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT_s)$, if $\Omega_s = \frac{2\pi}{T_s} \ge 2\Omega_N$.

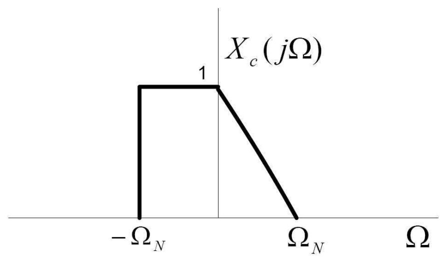


Figure 10.2: $X_c(j\Omega)$

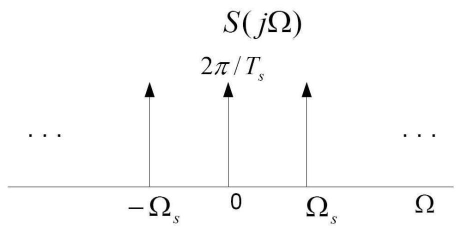


Figure 10.3: $S(j\Omega)$

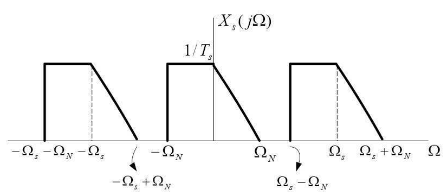


Figure 10.4: $X_s(j\Omega)$

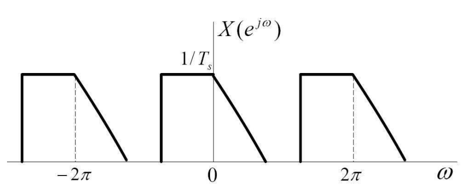


Figure 10.5: $X(e^{j\omega})$

This gives us an idea on how to reconstruct the original signal from the samples using (10.19). We use (10.19) to see that

$$\begin{aligned} x_r(t) &= h_r(t) * x_s(t) = \int_\tau h_r(t-\tau)x_s(\tau)\,\mathrm{d}\tau \\ &= \int_\tau \sum_n \delta(\tau - nT_s)x_c(nT_s)h_r(t-\tau)\,\mathrm{d}\tau \\ &= \sum_n x_c(nT_s)\int_\tau h_r(t-\tau)\delta(\tau-nT_s)\,\mathrm{d}\tau \end{aligned}$$

$$\longrightarrow x_r(t) = \sum_n x_c(nT_s)h_r(t-nT_s), \tag{10.21}$$

where $h_r(t)$ is the inverse CTFT of $H_r(j\Omega)$. The form of (10.21) shows the underlying operation as an *interpolating* between the sampled values. This point-of-view will be developed next in an alternate proof of the sampling theorem in terms of Hilbert spaces and bases functions. Finally note that since we choose $\Omega_s \geq 2\Omega_N$, we have perfect reconstruction, i.e.,

$$x_r(t) = x_c(t).$$

10.5 The Space of Bandlimited Signals.

The class of Ω_N-bandlimited functions of finite energy forms a Hilbert space, with the inner product defined in (10.2). An orthogonal basis for the space of Ω_N-bandlimited functions can be obtained easily from the prototypical bandlimited function, the sinc; indeed, consider the family

$$\varphi^{(n)}(t) = \mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right), \qquad n \in \mathbb{Z} \tag{10.22}$$

where, once again, $T_s = \pi/\Omega_N$. Note that we have $\varphi^{(n)}(t) = \varphi^{(0)}(t-nT_s)$ so that each basis function is simply a shifted version of the prototype basis function $\varphi^{(0)}$. Orthogonality can be easily proved as follows: first of all, because of the symmetry of the sinc function and the time-invariance of the convolution, we can write

$$\begin{aligned} \langle \varphi^{(n)}(t), \varphi^{(m)}(t)\rangle &= \langle \varphi^{(0)}(t-nT_s), \varphi^{(0)}(t-mT_s)\rangle \\ &= \langle \varphi^{(0)}(nT_s-t), \varphi^{(0)}(mT_s-t)\rangle \\ &= (\varphi^{(0)} * \varphi^{(0)})((n-m)T_s). \end{aligned}$$

We can now apply the convolution theorem and (10.9) to obtain

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{\Omega_N} \mathrm{rect}\left(\frac{\Omega}{2\Omega_N} \right) \right)^2 e^{j\Omega(n-m)T_s} d\Omega \\
&= \frac{\pi}{2\Omega_N^2} \int_{-\Omega_N}^{\Omega_N} e^{j\Omega(n-m)T_s} d\Omega \\
&= \begin{cases} \dfrac{\pi}{\Omega_N} = T_s & \text{if } n = m \\ \\ 0 & \text{if } n \neq m \end{cases}
\end{aligned}$$

so that $\{\varphi^{(n)}(t)\}_{n\in\mathbb{Z}}$ is orthogonal with normalization factor Ω_N/π (or, equivalently, $1/T_s$).

In order to show that the space of Ω_N-bandlimited functions is indeed a Hilbert space, we should also prove that the space is complete. This is a more delicate notion to show[5] and here it will simply be assumed.

10.5.1 Sampling as a Basis Expansion.

Now that we have an orthogonal basis, we can compute coefficients in the basis expansion of an arbitrary Ω_N-bandlimited function $x(t)$. We have

$$\begin{aligned}
\langle \varphi^{(n)}(t), x(t) \rangle &= \langle \varphi^{(0)}(t - nT_s), x(t) \rangle && (10.23) \\
&= (\varphi^{(0)} * x)(nT_s) && (10.24) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{\Omega_N} \mathrm{rect}\left(\frac{\Omega}{2\Omega_N} \right) X(j\Omega) e^{j\Omega n T_s} d\Omega && (10.25) \\
&= \frac{\pi}{\Omega_N} \; \frac{1}{2\pi} \int_{-\Omega_N}^{\Omega_N} X(j\Omega) e^{j\Omega n T_s} d\Omega && (10.26) \\
&= T_s \; x(nT_s) && (10.27)
\end{aligned}$$

in the derivation we have first rewritten the inner product as a convolution operation, then we have applied the convolution theorem, and recognized the penultimate line as simply the inverse CTFT of $X(j\Omega)$ calculated in $t = nT_s$. We therefore have the remarkable result that the n-th basis expansion coefficient is *proportional to the sampled value of* $x(t)$ at $t = nT_s$. For this reason, the sinc basis expansion is also called *sinc sampling.*

Reconstruction of $x(t)$ from its projections can now be achieved via the orthonormal basis reconstruction formula (4.40); since the sinc basis is just orthogonal rather than

[5]Completeness of the sinc basis can be proven as a consequence of the completeness of the Fourier basis in the continuous-time domain.

orthonormal, (4.40) needs to take into account the normalization factor and we have:

$$
\begin{aligned}
x(t) &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \langle \varphi^{(n)}(t), x(t) \rangle \, \varphi^{(n)}(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \mathrm{sinc}\left(\frac{t - nT_s}{T_s} \right)
\end{aligned}
\tag{10.28}
$$

which corresponds to the interpolation formula (10.37).

The Sampling Theorem. *If $x(t)$ is a Ω_N-bandlimited continuous-time signal, a* sufficient *representation of $x(t)$ is given by the discrete-time signal $x[n] = x(nT_s)$, with $T_s = \pi/\Omega_N$. The continuous time signal $x(t)$ can be exactly reconstructed from the discrete-time signal $x[n]$ as:*

$$
x(t) = \sum_{n=-\infty}^{\infty} x[n] \mathrm{sinc}\left(\frac{t - nT_s}{T_s} \right).
$$

A few notes:

- The proof of the theorem is inherent to the properties of the Hilbert space of bandlimited functions, and it is trivial after having proved the existence of an orthogonal basis.

- Clearly, if a signal is Ω_N-bandlimited, then it is also Ω-bandlimited for all $\Omega \geq \Omega_N$. Therefore, an Ω_N-bandlimited signal $x(t)$ is uniquely represented by all sequences $x[n] = x(nT)$ for which $T \leq T_s = \pi/\Omega_N$; T_s is the largest sampling period which guarantees perfect reconstruction (i.e., we cannot take fewer than $1/T_s$ samples per second).

- Another way to state the above point is to say that the *minimum* sampling frequency Ω_s for perfect reconstruction is exactly twice the Nyquist frequency, where the Nyquist frequency is the highest frequency of the bandlimited signal; the sampling frequency Ω must therefore satisfy the relationship:

  $$\Omega \geq \Omega_s = 2\Omega_N$$

  or, in Hertz,

  $$F \geq F_s = 2F_N.$$

10.5.2 Examples for the Sampling Theorem

We have seen that if a signal has a maximum frequency of $f_{\max}$, then sampling at a rate $f_s \geq 2f_{\max}$ is sufficient to retain all the information in the samples. Moreover, we can recover the original continuous-time signal from its samples using sinc interpolation.

Example 10.1 *Let* $x_c(t) = \cos(4000\pi t) = \cos[2\pi(2000)t]$, *for which the Fourier transform is shown in Fig. 10.6.*

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi).$$

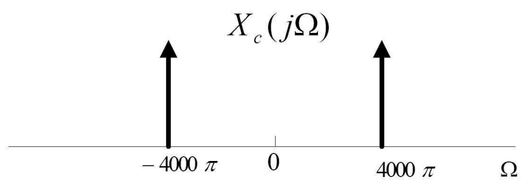


Figure 10.6: $X_c(j\Omega)$

Thus $f_{\max} = 2000$ *(*$\Omega_N = 4000\pi$*) for this case and we need* $f_s \geq 4000$ *as the sampling rate. Let* $f_s = \frac{1}{T_s} = 6000$, $\Omega_s = 2\pi f_s$.

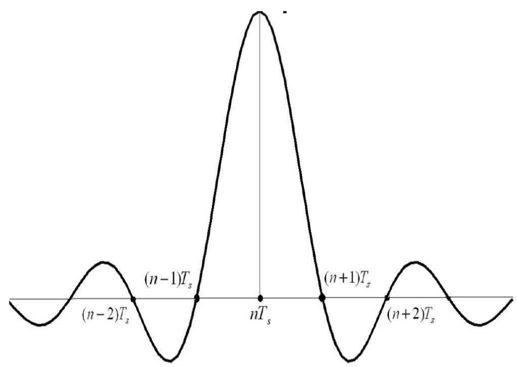
$$x[n] = x_c(nT_s) = \cos(2\pi 2000 nT_s) = \cos\left(2\pi\frac{2000}{6000}n\right) = \cos\left(\frac{2\pi}{3}n\right).$$

Now for reconstruction we get

$$\hat{x}_c(t) = \sum_{n=-\infty}^{+\infty} x[n]\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right) = \sum_{n=-\infty}^{+\infty} \cos\left(\frac{2\pi}{3}n\right)\frac{\sin\pi(6000t - n)}{\pi(6000t - n)}.$$

Let us look at this pictorially (Fig. 10.7).

$$\hat{X}_c(j\Omega) = H_r(j\Omega)X_s(j\Omega) = X_c(j\Omega).$$



Now, if $f_s = 1500 < 4000$*, then* $\Omega_s = 2\pi f_s = 3000\pi$*.*
But $\hat{x}_c(t) = \cos(1000\pi t) \neq \cos(4000\pi t) = x_c(t)$*.* $x[n] = \cos\frac{2\pi}{3}n$*, same as before! Fig. 10.8 shows the sampling and reconstruction in this case.*

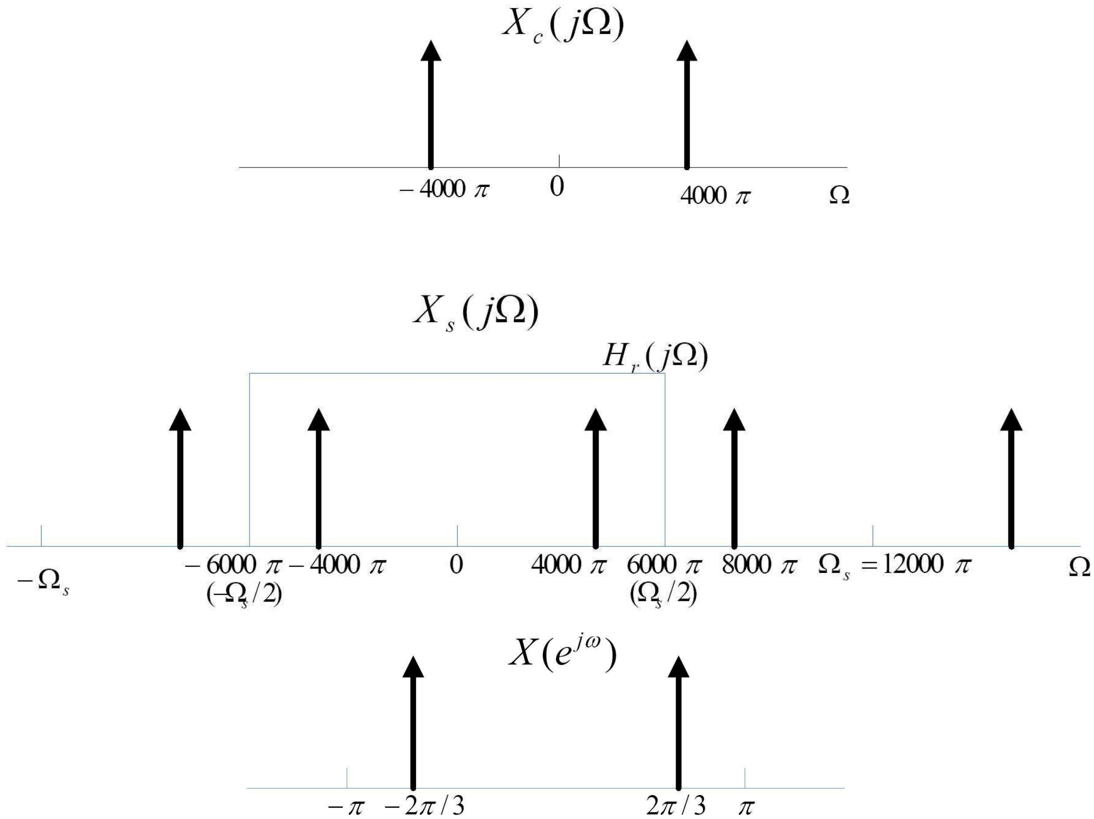


Figure 10.7: Pictorial representation of sampling of $x_c(t) = \cos(4000\pi t)$

Example 10.2 *Fig. 10.9 shows signal $X_c(j\Omega)$ and its sampled version.*

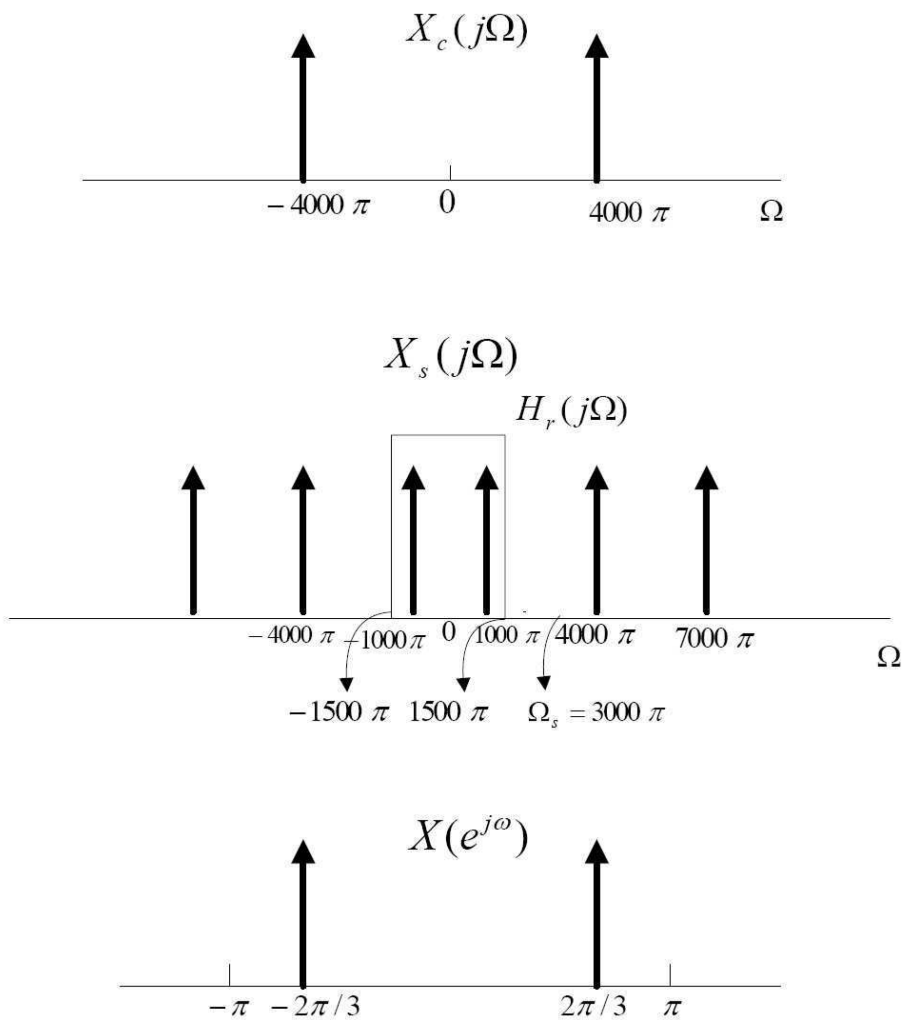


Figure 10.8: $X_c(j\Omega)$, $X_s(j\Omega)$, and $X\left(e^{j\omega}\right)$.

10.6 Interpolation

Interpolation is a procedure whereby we convert a discrete-time sequence $x[n]$ to a continuous-time function $x(t)$. Since this can be done in an arbitrary number of ways, we have to start by formulating some requirements on the resulting signal. At the heart of the interpolating

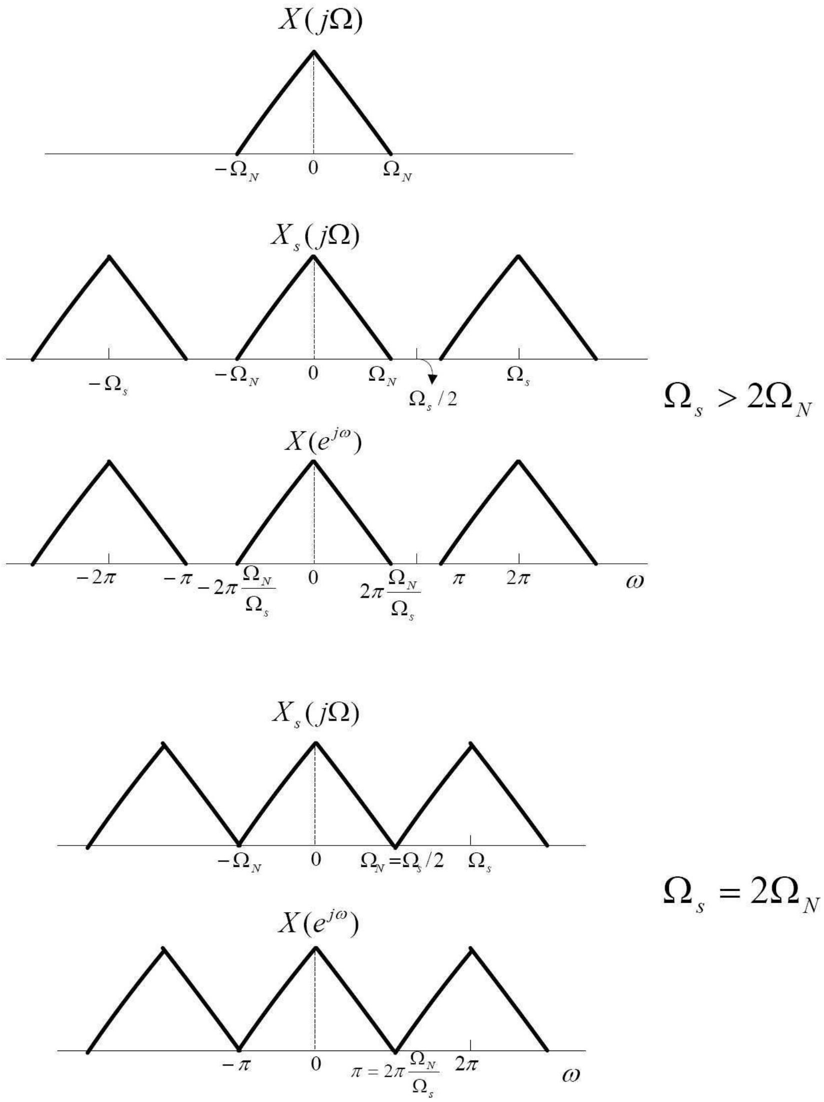


Figure 10.9: Example. 10.2

procedure, as we have mentioned, is the association of a physical time duration T_s to the interval between the samples in the discrete-time sequence. An intuitive requirement on the interpolated function is that its values at multiples of T_s be equal to the corresponding points of the discrete-time sequence, i.e.

$$x(t)|_{t=nT_s} = x[n];$$

the interpolation problem now reduces to “filling the gaps” between these instants.

10.6.1 Local Interpolation

The simplest interpolation schemes create a continuous-time function $x(t)$ from a discrete-time sequence $x[n]$ by setting $x(t)$ to be equal to $x[n]$ for $t = nT_s$ and by setting $x(t)$ to be some linear combination of neighboring sequence values when t lies in between interpolation instants. In general, the local interpolation schemes can be expressed by the following formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] I(\frac{t - nT_s}{T_s}), \tag{10.29}$$

where $I(t)$ is called the interpolation function (for linear functions the notation $I_N(t)$ is used and the subscript N indicates how many discrete-time samples besides the current one enter in the computation of the interpolated values for $x(t)$). The interpolation function must satisfy the fundamental *interpolation properties*:

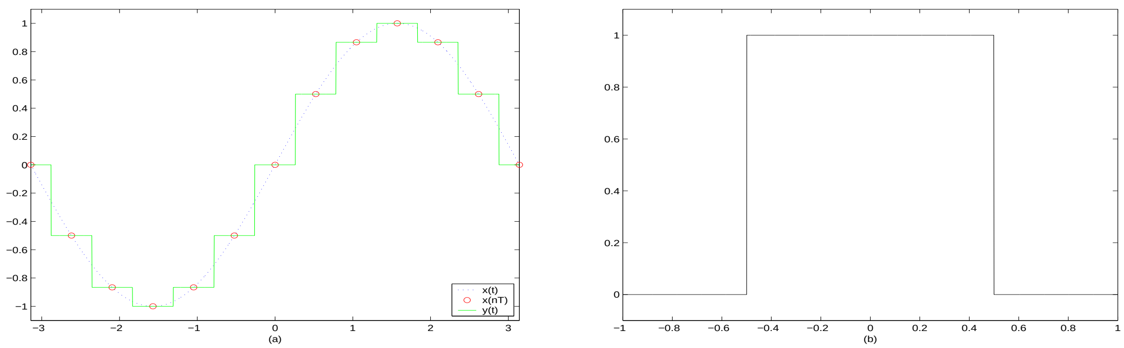
$$\begin{cases} I(0) = 1 \\ I(k) = 0 \text{ for } k \in \mathbb{Z} \setminus \{0\}, \end{cases} \tag{10.30}$$

where the second requirement implies that, no matter what the support of $I(t)$ is, its values should not affect other interpolation instants. By changing the function $I(t)$, we can change the type of interpolation and the properties of the interpolated signal $x(t)$.

Note that (10.29) can be interpreted either simply as a linear combination of shifted interpolation functions or, more interestingly, as a “mixed domain” convolution product, where we are convolving a discrete-time signal $x[n]$ with a continuous-time “impulse response” $I(t)$ scaled in time by the interpolation period T_s.

Zero-Order Hold. The simplest approach for the interpolating function is the piecewise-constant interpolation; here the continuous-time signal is kept constant between discrete sample values, yielding:

$$x(t) = x[n] \quad \text{for } (n - \frac{1}{2})T_s \leq t < (n + \frac{1}{2})T_s.$$



(a) (b)

Figure 10.10: Interpolation with zero-order hold. (a) Interpolation of the samples of a sinusoid. Note the discontinuities introduced by this simple scheme. (b) The rect function can be used to describe mathematically the zero-order hold.

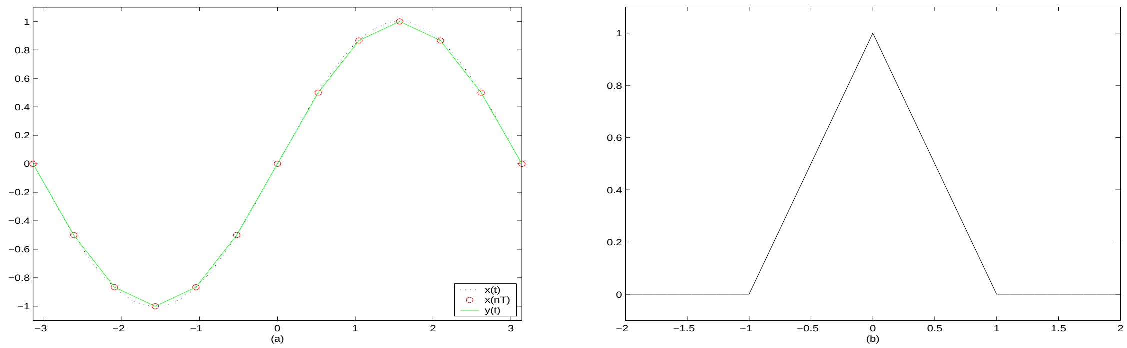
An example is shown in Figure 10.10(a); it is apparent that the resulting function is far from smooth since the interpolated function is discontinuous. The interpolation function is simply:

$$I_0(t) = \mathrm{rect}(t)$$

and the values of $x(t)$ depend only on the current discrete-time sample value.

First-Order Hold. A linear interpolator (sometimes called a first-order hold) simply connects the points corresponding to the samples with straight lines. An example is shown in Figure 10.11(a); note that now $x(t)$ depends on two consecutive discrete-time samples, across which a connecting straight line is drawn. From the point of view of smoothness, this interpolator already represents a good improvement over the zero-order hold: indeed the interpolated function is now continuous, although its first derivative is not. The first-order hold can be expressed in the same notation as in (10.29) by defining the following triangular function

$$I_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



(a) (b)

Figure 10.11: Linear interpolation (also called first-order hold). (a) Interpolation of the samples of a sinusoid using linear interpolation. (b) The triangular function is the interpolating function corresponding to the linear interpolation.

which is shown in Figure 10.11(b)[6]. It is immediate to verify that $I_1(t)$ satisfies the interpolation properties (10.30).

Higher-Order Interpolators. The zero- and first-order interpolators are widely used in practical circuits due to their extreme simplicity. These schemes can be extended to higher order interpolation functions and, in general, $I_N(t)$ will be an N-th order polynomial in t. The advantage of the local interpolation schemes is that, for small N, they can be easily implemented in practice as *causal* interpolation schemes (locality is akin to FIR filtering); their disadvantage is that, because of the locality, their N-th derivative will be discontinuous. This discontinuity represents a lack of smoothness in the interpolated function; from a spectral point of view this corresponds to a high frequency energy content, which is usually undesirable.

10.6.2 Polynomial Interpolation

The lack of smoothness of local interpolations is easily eliminated when we need to interpolate just a *finite* number of discrete-time samples. In fact, in this case the task becomes a classic polynomial interpolation problem for which the optimal solution has been known

[6]Note that $I_1(t) = (I_0 * I_0)(t)$.

for a long time under the name of *Lagrange interpolation*. Note that a polynomial interpolating a finite set of samples is a maximally smooth function in the sense that it is continuous together with all its derivatives.

Consider a length $(2N+1)$ discrete-time signal $x[n]$, with n$= -N, \ldots, N$. Associate to each sample an abscissa $t_n = nT_s$; we know from basic algebra that there is one and only one polynomial $P(t)$ of degree $2N$ which passes through all the $2N+1$ pairs $(t_n, x[n])$ and this polynomial is the Lagrange interpolator. The coefficients of the polynomial could be found by solving the set of $2N+1$ equations:

$$\{P(t_n) = x[n]\}_{n=-N,\ldots,N} \tag{10.31}$$

but a simpler way to determine the expression for $P(t)$ is to use the set of $2N+1$ *Lagrange polynomials* of degree $2N$:

$$\begin{aligned} L_n^{(N)}(t) &= \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{(t-t_k)}{(t_n-t_k)} \\ &= \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{t/T_s-k}{n-k} \qquad n=-N,\ldots,N. \end{aligned} \tag{10.32}$$

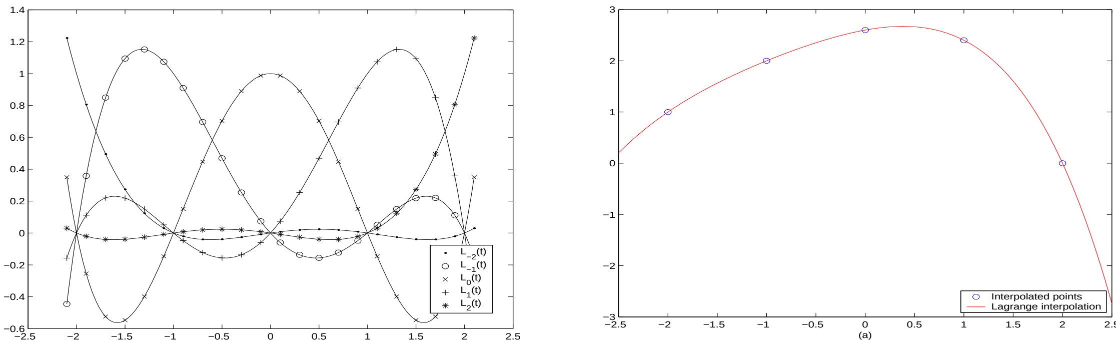
The polynomials $L_n^{(N)}(t)$ for $T_s = 1$ and $N = 2$ (i.e. interpolation of 5 points) are plotted in Figure 10.12-(a). By using this notation, the *global* Lagrange interpolator for a given set of abscissa/ordinate pairs can now be written as a simple linear combination of Lagrange polynomials:

$$P(t) = \sum_{n=-N}^{N} x[n] L_n^{(N)}(t) \tag{10.33}$$

and it is easy to verify that this is the unique interpolating polynomial of degree $2N$ in the sense of (10.31). Note that *each* of the $L_n^{(N)}(t)$ satisfies the interpolation properties (10.30) or, concisely (for $T_s = 1$):

$$L_n^{(N)}(m) = \delta[n-m], \; m \in \{-N,\ldots,N\}.$$

The interpolation formula, however, cannot be written in the form of (10.29) since the Lagrange polynomials are not simply shifts of a single prototype function. The continuous time signal $x(t) = P(t)$ is now a *global* interpolating function for the finite-length discrete-time signal $x[n]$, in the sense that it depends on *all* samples in the signal; as a consequence, $x(t)$ is maximally smooth ($x(t) \in C^\infty$). An example of Lagrange interpolation for $N = 2$ is plotted in Figure 10.12-(b).



(a) (b)

Figure 10.12: Lagrange interpolation. (a) The polynomials $L_n^{(N)}(t)$ used to compute the interpolation for $N = 2$ and $T = 1$. Note that $L_n^{(N)}(m)$ is zero except for $m = n$, where it is 1. (b) Interpolation using 5 points.

10.6.3 Sinc Interpolation

The beauty of local interpolation schemes lies in the fact that the interpolated function is simply a linear combination of shifted versions of the *same* prototype interpolation function $I(t)$; this unfortunately has the disadvantage of creating a continuous-time function which lacks smoothness. Polynomial interpolation, on the other hand, is perfectly smooth but it only works in the finite-length case and it requires different interpolation functions with different signal lengths. Yet, both approaches can come together in a nice mathematical way and we are now ready to introduce the maximally smooth interpolation scheme for infinite discrete-time signals.

Let us take the expression for the Lagrange polynomial of degree N in (10.32) and consider its limit for N going to infinity. We have:

$$
\begin{aligned}
\lim_{N\to\infty} L_n^{(N)}(t) &= \prod_{\substack{k=-\infty \\ k\neq n}}^{\infty} \frac{t/T_s - k}{n-k} \\
&= \prod_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{t/T_s - n + m}{m} \\
&= \prod_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(1 + \frac{t/T_s - n}{m}\right) \\
&= \prod_{m=1}^{\infty} \left(1 - \left(\frac{t/T_s - n}{m}\right)^2\right) \qquad (10.34) \\
& \qquad (10.35)
\end{aligned}
$$

where we have used the change of variable $m = n - k$. We can now invoke Euler's infinite product expansion for the sine function

$$
\sin(\pi\tau) = (\pi\tau)\prod_{k=1}^{\infty}\left(1 - \frac{\tau^2}{k^2}\right)
$$

(whose derivation is in the appendix) to finally obtain

$$
\lim_{N\to\infty} L_n^{(N)}(t) = \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right). \qquad (10.36)
$$

The convergence of the Lagrange polynomial $L_0^{(N)}(t)$ to the sinc function is illustrated in Figure 10.13. Note that now, as the number of points becomes infinite, the Lagrange polynomials converge to shifts of the *same* prototype function, i.e. the sinc; therefore the interpolation formula can be expressed as in (10.29) with $I(t) = \operatorname{sinc}(t)$; indeed, if we consider an infinite sequence $x[n]$ and apply the Lagrange interpolation formula (10.33) we obtain:

$$
x(t) = \sum_{n=-\infty}^{\infty} x[n]\operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right). \qquad (10.37)
$$

Spectral Properties of the Sinc Interpolation. The sinc interpolation of a discrete-time sequence gives rise to a strictly bandlimited continuous-time function. If the DTFT $X(e^{j\omega})$

of the discrete-time sequence exists, the spectrum of the interpolated function $X(j\Omega)$ can be obtained as follows:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\text{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j\Omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \text{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j\Omega t} dt \end{aligned}$$

now we use (10.9) to get the Fourier Transform of the scaled and shifted sinc

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \text{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$

and use the fact that, as usual, $T_s = \pi/\Omega_N$

$$\begin{aligned} &= \left(\frac{\pi}{\Omega_N}\right) \text{rect}\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi(\Omega/\Omega_N)n} \\ &= \begin{cases} (\pi/\Omega_N) X(e^{j\pi\Omega/\Omega_N}) & \text{for } |\Omega| \leq \Omega_N \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, the continuous-time spectrum is just a scaled and stretched version of the DTFT of the discrete-time sequence between $-\pi$ and π. The duration of the interpolation interval T_s is inversely proportional to the resulting bandwidth of the interpolated signal. Intuitively, a slow interpolation (T_s large) will result in a spectrum concentrated around the low frequencies; conversely, a fast interpolation (T_s small) will result in a spread-out spectrum (more high frequencies are present)[7].

10.7 Aliasing

The "naive" notion of sampling, as we have seen, is associated to the very practical idea of measuring the instantaneous value of a continuous-time signal at uniformly spaced instants in time. For bandlimited signals, we have seen that this is actually equivalent to an orthogonal decomposition in the space of bandlimited functions, which guarantees that the set of samples $x(nT_s)$ uniquely determines the signal and allows its perfect reconstruction. We now want to address the following question: what happens if we simply sample an *arbitrary* time signal in the "naive" sense (i.e. in the sense of simply taking $x[n] = x(nT_s)$) and what can we reconstruct from the set of samples thus obtained?

[7] To find a simple everyday analogy, think of a 45rpm vinyl record played at either 33rpm (slow interpolation) or at 78rmp (fast interpolation) and remember the acoustic effect on the sounds.

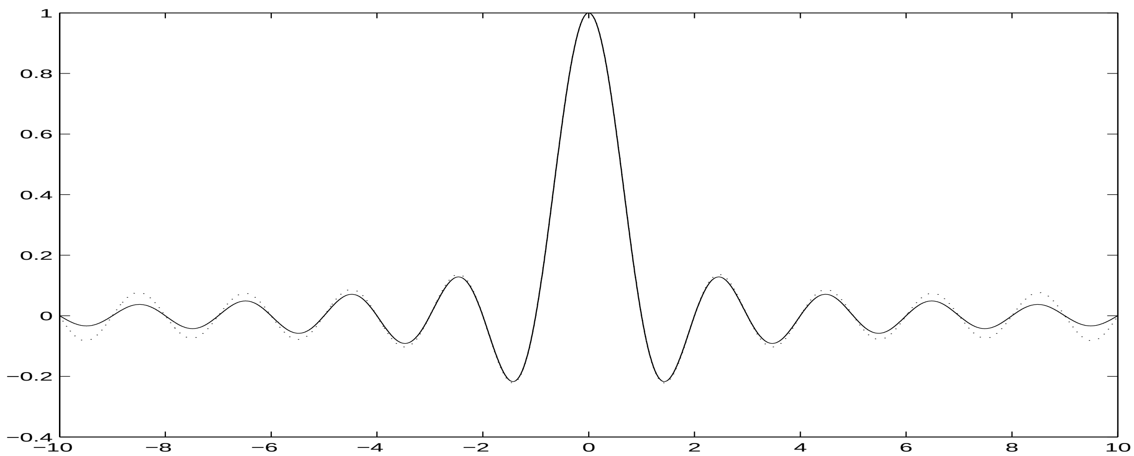


Figure 10.13: The sinc function (solid) and its Lagrange approximation (dashed) as in (10.34) for 100 factors in the product.

10.7.1 Non-Bandlimited Signals

Given a sampling period of T_s seconds, the sampling theorem ensures that there is no loss of information by sampling the class of Ω_N-bandlimited signals, where as usual $\Omega_N = \pi/T_s$. If a signal $x(t)$ is not Ω_N-bandlimited (i.e. its spectrum is nonzero at least somewhere outside of $[-\Omega_N, \Omega_N]$) then the approximation properties of orthogonal bases state that its *best* approximation in terms of uniform samples T_s seconds apart is given by the samples *of its projection over the space of* Ω_N*-bandlimited signals*. This is easily seen in (10.26), where the projection is easily recognizable as an ideal lowpass filtering operation on $x(t)$ (with gain T_s) which truncates its spectrum outside of the $[-\Omega_N, \Omega_N]$ interval.

Sampling as the result of a sinc basis expansion automatically includes this lowpass filtering operation; for a Ω_N-bandlimited signal, obviously, the filtering is just a scaling by T_s. For an arbitrary signal, however, we can now decompose the sinc sampling as in Figure 10.14, where the first block is a continuous-time lowpass filter with cutoff frequency Ω_N and gain $T_s = \pi/\Omega_N$. The discrete time sequence $x[n]$ thus obtained is the best discrete-time approximation of the original signal when the sampling is uniform.

10.7.2 Aliasing: Intuition

Now let's go back to the naive sampling scheme in which simply $x[n] = x(nT_s)$, with $F_s = 1/T_s$ the sampling frequency of the system; what is the error we incur if $x(t)$ is

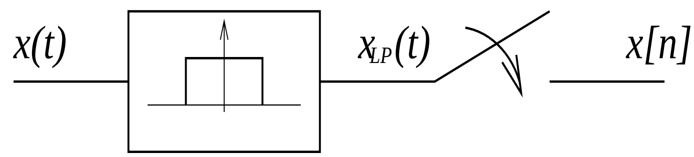


Figure 10.14: Bandlimited sampling (sinc basis expansion) as a combination of lowpass filtering (in the continuous-time domain) and sampling; $x_{LP}(t)$ is the projection of $x(t)$ over the space of Ω_N-bandlimited functions.

not bandlimited or, equivalently, if the sampling frequency is less than twice the Nyquist frequency? We will develop the intuition by starting with the simple case of a single sinusoid and we will move on to a formal demonstration of the aliasing phenomenon. In the following examples we will work with frequencies in Hertz, both out of practicality and to give an example of a different form of notation.

Sampling of Sinusoids. Consider a simple continuous-time signal such as $x(t) = e^{j2\pi f_0 t}$ and its sampled version $x[n] = e^{j2\pi(f_0/F_s)n} = e^{j\omega_0 n}$ with

$$\omega_0 = 2\pi \frac{f_0}{F_s}. \tag{10.38}$$

Clearly, since $x(t)$ contains only one frequency, it is Ω-bandlimited for all $\Omega > 2\pi|f_0|$. If the frequency of the sinusoid satisfies $|f_0| < F_s/2 = F_N$, then $\omega_0 \in (-\pi, \pi)$ and the frequency of the original sinusoid can be univocally determined from the sampled signal. Now assume that $f_0 = F_N = F_s/2$; we have

$$x[n] = e^{j\pi n} = e^{-j\pi n}.$$

In other words, we encounter a first ambiguity with respect to the direction of rotation of the complex exponential: from the sampled signal we cannot determine whether the original frequency was $f_0 = F_N$ or $f_0 = -F_N$. If we increase the frequency further, say $f_0 = (1+\alpha)F_N$, we have

$$x[n] = e^{j(1+\alpha)\pi n} = e^{-j\alpha\pi n}.$$

Now the ambiguity is both on the direction and on the frequency value: if we try to infer the original frequency from the sampled sinusoid from (10.38) we cannot discriminate between $f_0 = (1+\alpha)F_N$ or $f_0 = -\alpha F_N$. Matters get even worse if $f_0 > F_s$. Suppose we can write $f_0 = F_s + f_b$ with $f_b < F_s/2$; we have

$$x[n] = e^{j(2\pi F_s T_s + 2\pi f_b T_s)n} = e^{j(2\pi + \omega_b)n} = e^{j\omega_b n},$$

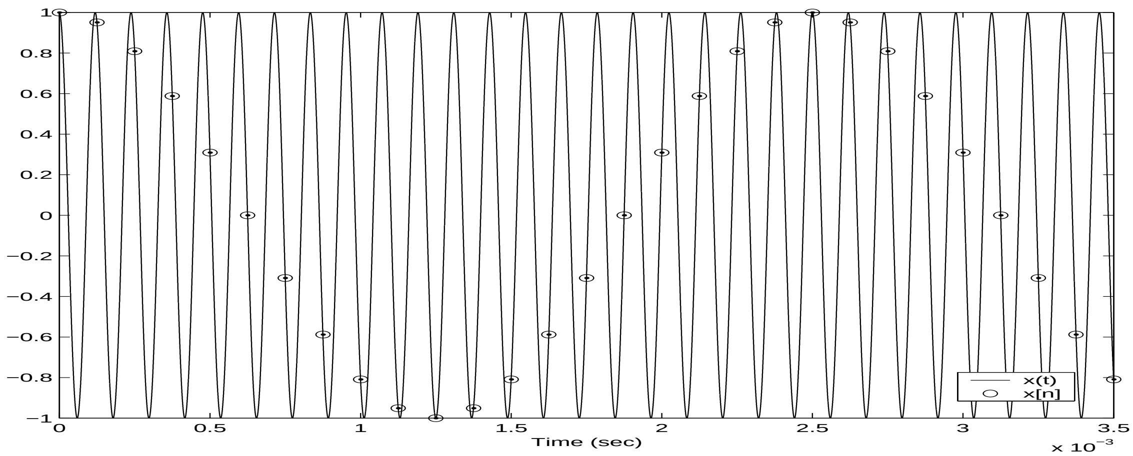


Figure 10.15: Example of aliasing: complex sinusoid at 8400 Hz, $x(t) = e^{j(2\pi \cdot 8400)t}$; sampling frequency $F_s = 8000$ Hz. The sampled signal is indistinguishable from a sinusoid at 400 Hz sampled at F_s (in the plot, only the real part is shown).

so that the sinusoid is completely undistinguishable from a sinusoid of frequency f_b sampled at F_s; the fact that two continuous-time frequencies are mapped to the same discrete-time frequency is called *aliasing*. An example of aliasing is depicted in Figure 10.15.

In general, because of the 2π-periodicity of the discrete-time complex exponential, we can always write

$$\omega_b = (2\pi f_0 T_s) + 2k\pi$$

and choose $k \in \mathbb{Z}$ so that ω_b falls in the $[-\pi, \pi]$ interval. Seen the other way, all continuous-time frequencies of the form

$$f = f_b + kF_s$$

with $f_b < F_N$ are aliased to the same discrete-time frequency ω_b.

Consider now the signal $y(t) = Ae^{j2\pi f_b t} + Be^{j2\pi(f_b+F_s)t}$, with $f_b < F_N$. If we sample this signal with sampling frequency F_s we obtain

$$\begin{aligned} x[n] &= Ae^{j2\pi(f_b/F_s)n} + Be^{j2\pi(f_b/F_s+1)nT_s} \\ &= Ae^{j\omega_b n} + Be^{j\omega_b n}e^{j2\pi n} \\ &= (A+B)e^{j\omega_b n} \end{aligned}$$

In other words, two continuous-time exponential which are F_s Hz apart will give rise to a single discrete-time complex exponential, whose amplitude is equal to the sum of the amplitudes of both the original sinusoids.

Energy Folding of the Fourier Transform. To understand what happens to a general signal, consider the interpretation of the Fourier transform as a bank of (infinitely many) complex oscillators initialized with phase and amplitude, each contributing to the energy content of the signal at their respective frequency. Since in the sampled version any two frequencies F_s apart are undistinguishable, their contributions to the discrete-time Fourier transform of the sampled signal will add up. This aliasing can be represented as a spectral *superposition*: the continuous-time spectrum above F_N is cut, shifted back to $-F_N$, summed over $[-F_N, F_N]$, and the process is repeated again and again; the same for the spectrum below $-F_N$. This process is nothing but the familiar periodization of a signal:

$$\sum_{k=-\infty}^{\infty} X(j2\pi f + j2k\pi F_s)$$

as we will prove formally in the next section.

10.7.3 Aliasing: Proof

In the following we will consider the relationship between the DTFT of a sampled signal $x[n]$ and the continuous-time Fourier transform (CTFT) of the originating continuous-time signal $x_c(t)$. For clarity, we will add the subscript $(\cdot)_c$ to all continuous-time quantities so that, for instance, we will write $x[n] = x_c(nT_s)$.

Consider $X(e^{j\omega})$, the DTFT of the sampled sequence (with, as usual, $T_s = (1/F_s) = (\pi/\Omega_N)$). The inversion formula states:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{10.39}$$

We can arrive at an expression for $x[n]$ also from $X_c(j\Omega)$, the Fourier transform of the continuous-time function $x_c(t)$; indeed:

$$x[n] = x_c(nT_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega\, nT_s}d\Omega. \tag{10.40}$$

The idea is to split the integration interval in the above expression as the sum of non overlapping intervals whose width is equal to the sampling bandwidth $\Omega_s = 2\Omega_N$; this stems from the realization that, in the inversion process, all frequencies Ω_s apart will give undistinguishable contribution to the discrete-time spectrum. We have:

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\Omega_N}^{(2k+1)\Omega_N} X_c(j\Omega)e^{j\Omega\, nT_s}d\Omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\Omega_N}^{\Omega_N} X_c(j\Omega - jk\Omega_s)e^{j\Omega\, nT_s}d\Omega \qquad (10.41)\\
&= \frac{1}{2\pi}\int_{-\Omega_N}^{\Omega_N}\left\{\sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s)\right\}e^{j\Omega\, nT_s}d\Omega \qquad (10.42)\\
&= \frac{1}{2\pi}\int_{-\Omega_N}^{\Omega_N}\tilde{X}_c(j\Omega)e^{j\Omega\, nT_s}d\Omega \qquad (10.43)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{T_s}\tilde{X}_c(j\frac{\theta}{T_s})e^{j\theta n}d\theta \qquad (10.44)
\end{aligned}$$

A few notes on the above derivation:

(a) In (10.41) we have exploited the Ω_s-periodicity of $e^{j\Omega\, nT_s}$ (i.e. $e^{j(\Omega+k\Omega_s)nT_s} = e^{j\Omega nT_s}$).

(b) In (10.42) we have interchanged the order of integration and summation. This can be done under fairly broad conditions on $x_c(t)$, for which we refer to the bibliography.

(c) In (10.43) we have defined

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s)$$

which is just the periodized version of $X_c(j\Omega)$.

(d) In (10.44) we have operated the change of variable $\theta = \Omega T_s$. It is immediate to verify that $\tilde{X}_c(j(\theta/T_s))$ is now 2π-periodic in θ.

If we now compare (10.44) to (10.39) we can easily see that (10.44) is nothing but the DTFT inversion formula for the 2π-periodic function $(1/T_s)\tilde{X}(j\theta/T_s)$; since the inversion formulas (10.44) and (10.39) yield the same result (namely, $x[n]$) we can conclude that

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}) \qquad (10.45)$$

which is the relationship between the Fourier transform of a continuous-time function and the DTFT of its sampled version, with T_s being the sampling period. The above result is a particular version of a more general result in Fourier theory called the *Poisson sum formula*. In particular, when $x_c(t)$ is Ω_N-bandlimited, the copies in the periodized spectrum do not overlap and the (periodic) discrete-time spectrum between $-\pi$ and π is simply

$$X(e^{j\omega}) = \frac{1}{T_s} X_c(j\frac{\omega}{T_s}).$$

10.7.4 Examples

Figures 10.16 to 10.19 illustrate several examples of the relationship between the continuous-time spectrum and the discrete-time spectrum. For all figures, the top panel shows the continuous-time spectrum, with labels indicating the Nyquist frequency (where applicable) and the sampling frequency. In particular:

- Figure 10.16 shows the result of sampling a bandlimited signal with a sampling frequency in excess of the minimum (twice the Nyquist frequency); in this case we say that the signal has been *oversampled*. The result is that in the periodized spectrum the copies do not overlap and the discrete-time spectrum is just a scaled version of the original spectrum (with even a narrower support than the full $[-\pi, \pi]$ range because of the oversampling).

- Figure 10.17 shows the result of sampling a bandlimited signal with a sampling frequency exactly equal to twice the Nyquist frequency; in this case we say that the signal has been *critically sampled*. In the periodized spectrum the copies again do not overlap and the discrete-time spectrum is a scaled version of the original spectrum.

- Figure 10.18 shows the result of sampling a bandlimited signal with a sampling frequency less than the minimum sampling frequency. Now in the periodized spectrum the copies do overlap, and the resulting discrete-time spectrum is an aliased version of the original; the original spectrum cannot be reconstructed from the sampled signal.

- Finally, Figure 10.19 shows the result of sampling a non-bandlimited signal with a sampling frequency which is chosen as a tradeoff between alias and number of samples per second. The idea is to disregard the low-energy "tails" of the original spectrum so that their alias does not corrupt too much the discrete-time spectrum. In the periodized spectrum the copies do overlap and the resulting discrete-time spectrum is an aliased version of the original, which is similar to the original; the original spectrum, however, cannot be reconstructed from the sampled signal. In a practical sampling scenario, the correct design choice would have been to lowpass filter (in the continuous-time domain) the original signal so as to eliminate the spectral tails beyond $\pm\Omega_s/2$.

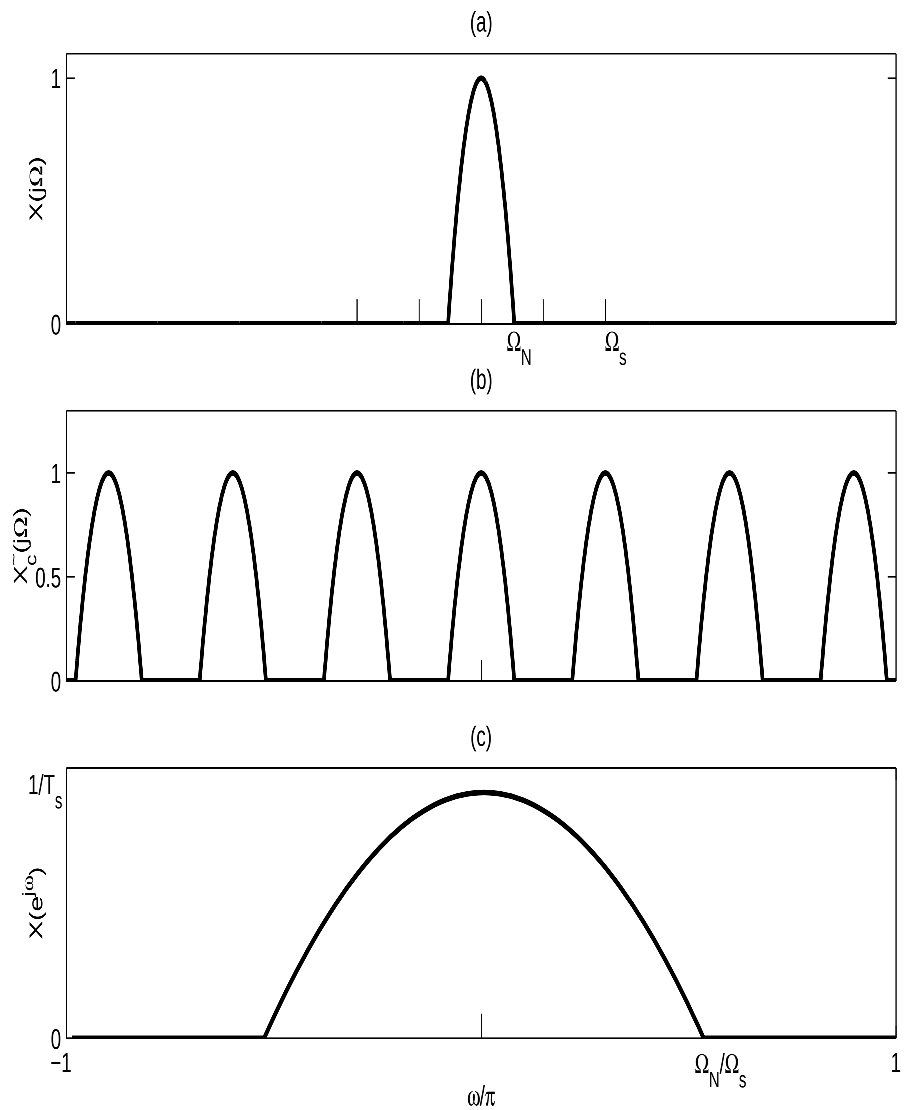


Figure 10.16: Sampling of a bandlimited signal – Case 1: $\Omega_s > 2\Omega_N$.
(a) Original continuous-time spectrum $X_c(j\Omega)$; (b) Periodized spectrum (thick line) $\tilde{X}_c(j\Omega)$; (c) Discrete-time spectrum $X(e^{j\omega})$ in the interval $[-\pi, \pi]$.

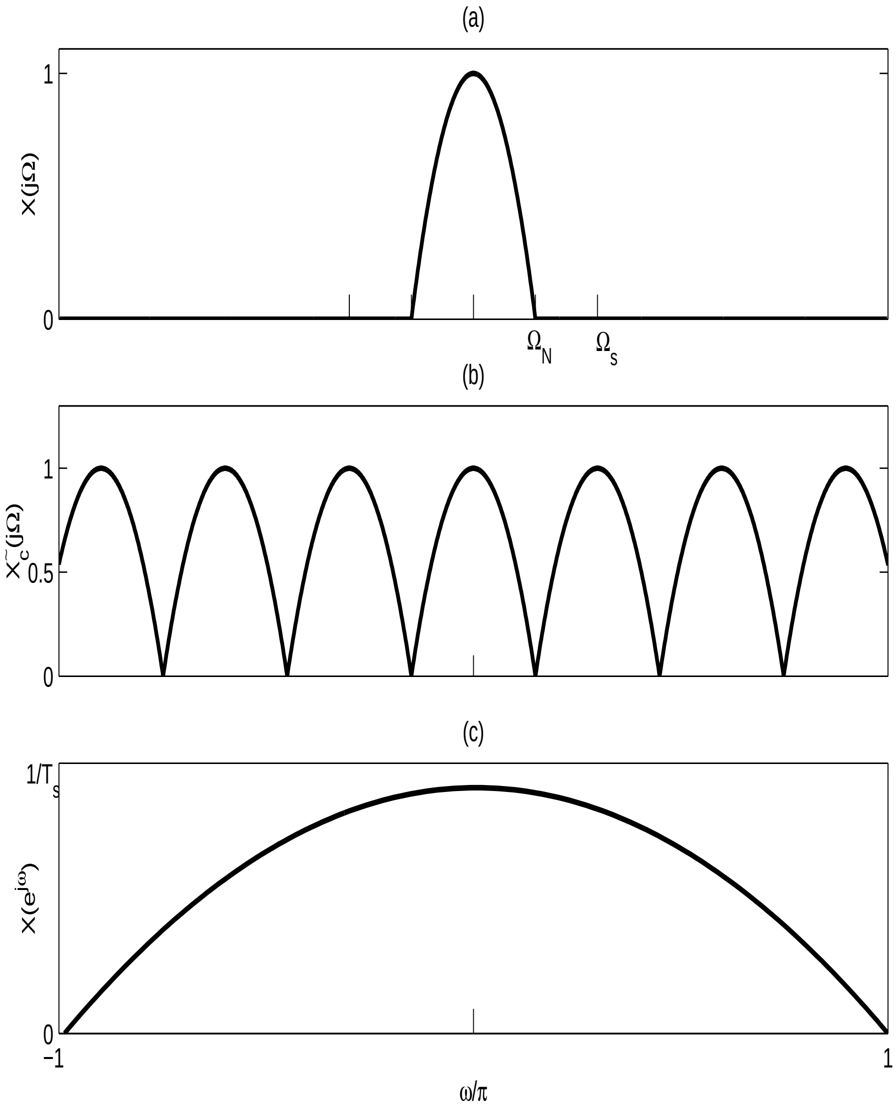


Figure 10.17: Sampling of a bandlimited signal – Case 2: $\Omega_s = 2\Omega_N$.

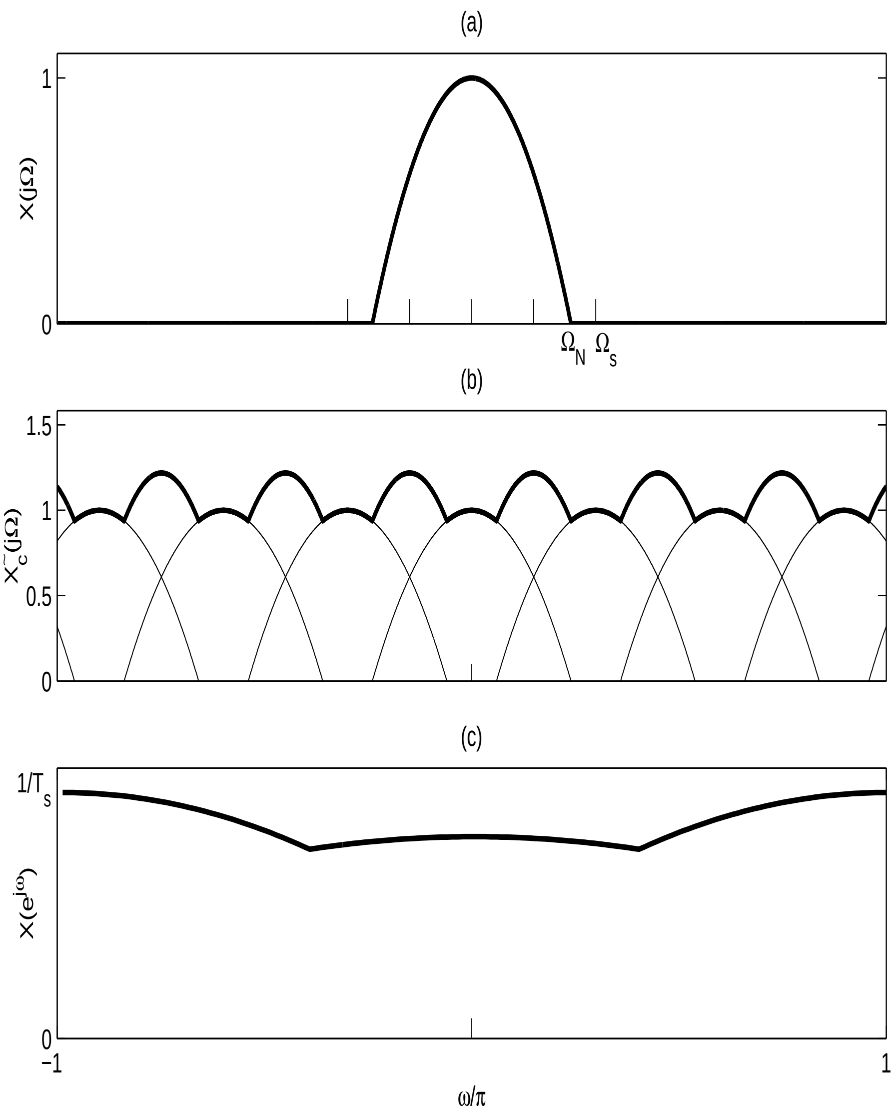


Figure 10.18: Sampling of a bandlimited signal – Case 3: $\Omega_s < 2\Omega_N$ (aliasing).

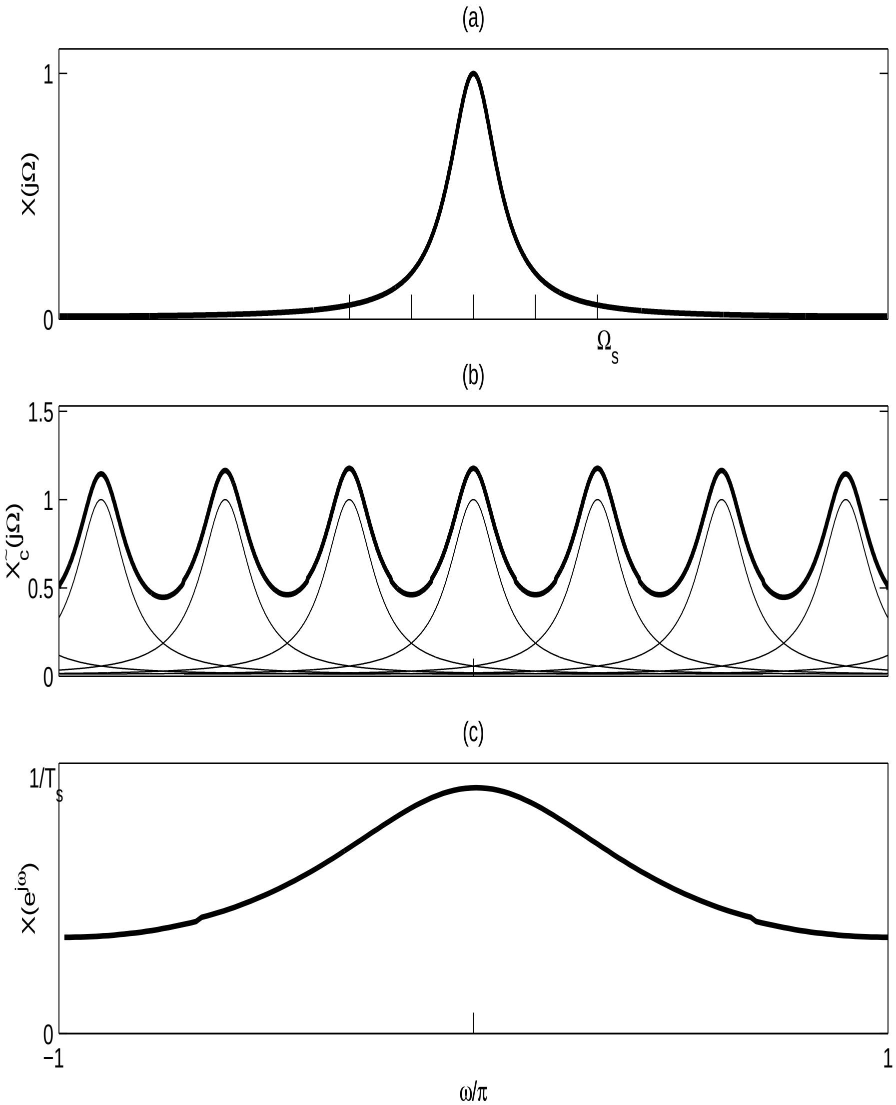


Figure 10.19: Sampling of a non-bandlimited signal.

Example 10.3 *Consider*

$$x_c(t) = e^{j2\pi f_0 t}$$

with corresponding sampled version

$$x[n] = x_c(nT_s) = e^{j2\pi f_0 \frac{n}{f_s}} = e^{j2\pi\left(\frac{f_0}{f_s}\right)n} = e^{j\omega_0 n},$$

where

$$\omega_0 = 2\pi\left(\frac{f_0}{f_s}\right).$$

Let $f_0 = \frac{f_s}{2}$*, then* $\omega_0 = \pi$ *and*

$$x[n] = e^{j\pi n} = (-1)^n = e^{-j\pi n}.$$

Hence we have an ambiguity; from the sampled signal we cannot tell if the original signal was $e^{j2\pi f_0 t}$ *or* $e^{-j2\pi f_0 t}$ *(Fig. 10.20).*

Example 10.4 *Consider the continuous-time signal*

$$x_a(t) = \cos(2\pi F_0 t)$$

(a) Compute analytically the spectrum $X_a(F)$ *of* $x_a(t)$*. (Hint:* $e^{jat} \overset{\mathcal{F}}{\leftrightarrow} \delta(f - \frac{a}{2\pi})$*)*

(b) Compute analytically the spectrum of the signal $x[n] = x_a(nT)$*,* $T = \frac{1}{F_s}$*.*

(c) Plot the magnitude spectrum $|X_a(F)|$ *for* $F_0 = 10$ *Hz.*

(d) Plot the magnitude spectrum $|X(e^{j\omega})|$ *for* $F_s = 10, 20, 40$ *and* 100 *Hz.*

(e) Explain the results obtained in the previous part in terms of aliasing effects.

Solution:

(a) It can easily be seen that $X_a(F) = \frac{1}{2}(\delta(F - F_0) + \delta(F + F_0))$*. Indeed,*

$$\begin{aligned} X_a(F) &= \int \cos(2\pi F_0 t) e^{-j2\pi F t} dt \\ &= \frac{1}{2}\left[\int e^{j2\pi F_0 t} e^{-j2\pi F t} dt + \int e^{-j2\pi F_0 t} e^{-j2\pi F t} dt\right] \\ &= \frac{1}{2}(\delta(F - F_0) + \delta(F + F_0)) \end{aligned}$$

using the hint.

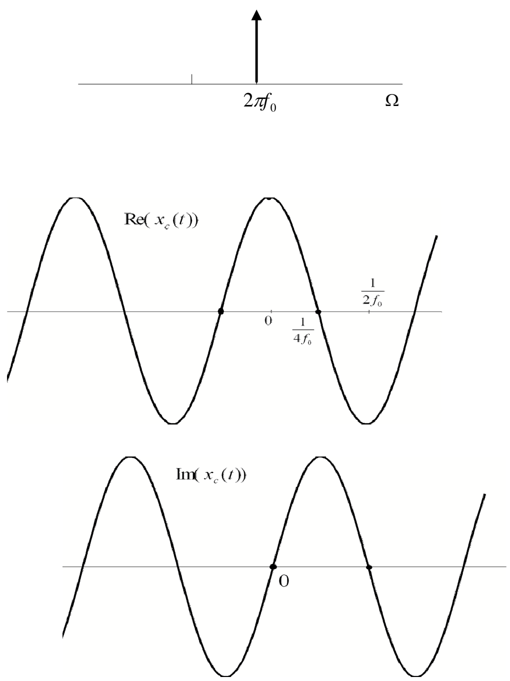


Figure 10.20: Example. 10.3

(b) *We know that* $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(e^{j\frac{\omega}{T}-j\frac{2\pi k}{T}})$. *Further, we know that* $\omega = 2\pi\frac{F}{F_s}$. *Hence,* $X(F) = F_s\sum_{k=-\infty}^{\infty} X_a(F - kF_s) = \frac{F_s}{2}\sum_{k=-\infty}^{\infty}[\delta(F - F_0 - kF_s) + \delta(F + F_0 - kF_s)]$. *This means that the spectrum of the continuous time signal is repeated every* F_s *when we sample it.*

(c) *See Fig. 10.21.*

(d) *See Fig. 10.22.*

(e) *When the sampling frequency is less than or equal to 2 times* F_0, i.e., *when* $F_s \leq 2F_0$,

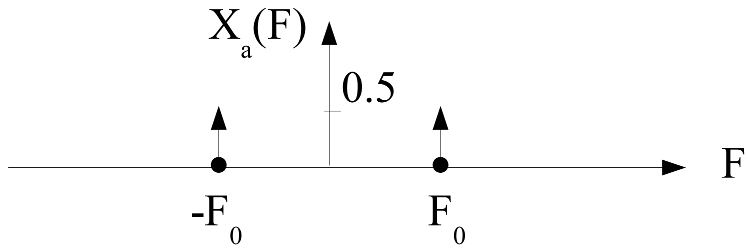


Figure 10.21: Spectrum of $X_a(F)$

then we can see that the signal is aliased.

10.8 Problems

Problem 10.1 *Consider a real function $f(t)$ for which the Fourier transform is well defined:*

$$F(j\Omega) = \int_{-\infty}^{\infty} f(t)e^{-j\Omega t}dt\,. \tag{10.46}$$

Suppose that we only possess a discrete-time version of $f(t)$, that is, we only know the values of $f(t)$ at times $t = n\Delta, n \in \mathbb{Z}$ for a fixed interval Δ. We want to approximate $F(j\Omega)$ with the following expression:

$$\hat{F}(j\Omega) = \sum_{n=-\infty}^{\infty} \Delta \cdot f(n\Delta)e^{-j\Delta n\Omega}\,. \tag{10.47}$$

Observe that $F(j\Omega)$ in (10.46) is computed using the values of $f(t)$ for all t, while the approximation in (10.47) uses only the values of $f(t)$ for a countable number of t.

Consider now the periodic repetition *of $F(j\Omega)$:*

$$\tilde{F}(j\Omega) = \sum_{n=-\infty}^{\infty} F(j(\Omega + \frac{2\pi}{\Delta}n))\,. \tag{10.48}$$

That is, $F(j\Omega)$ is repeated (with possible overlapping) with period $2\pi/\Delta$ (same Δ as in the approximation (10.47)).

(a) *Show that the approximation $\hat{F}(j\Omega)$ is equal to the periodic repetition of $F(j\Omega)$,* i.e.,

$$\hat{F}(j\Omega) = \tilde{F}(j\Omega).$$

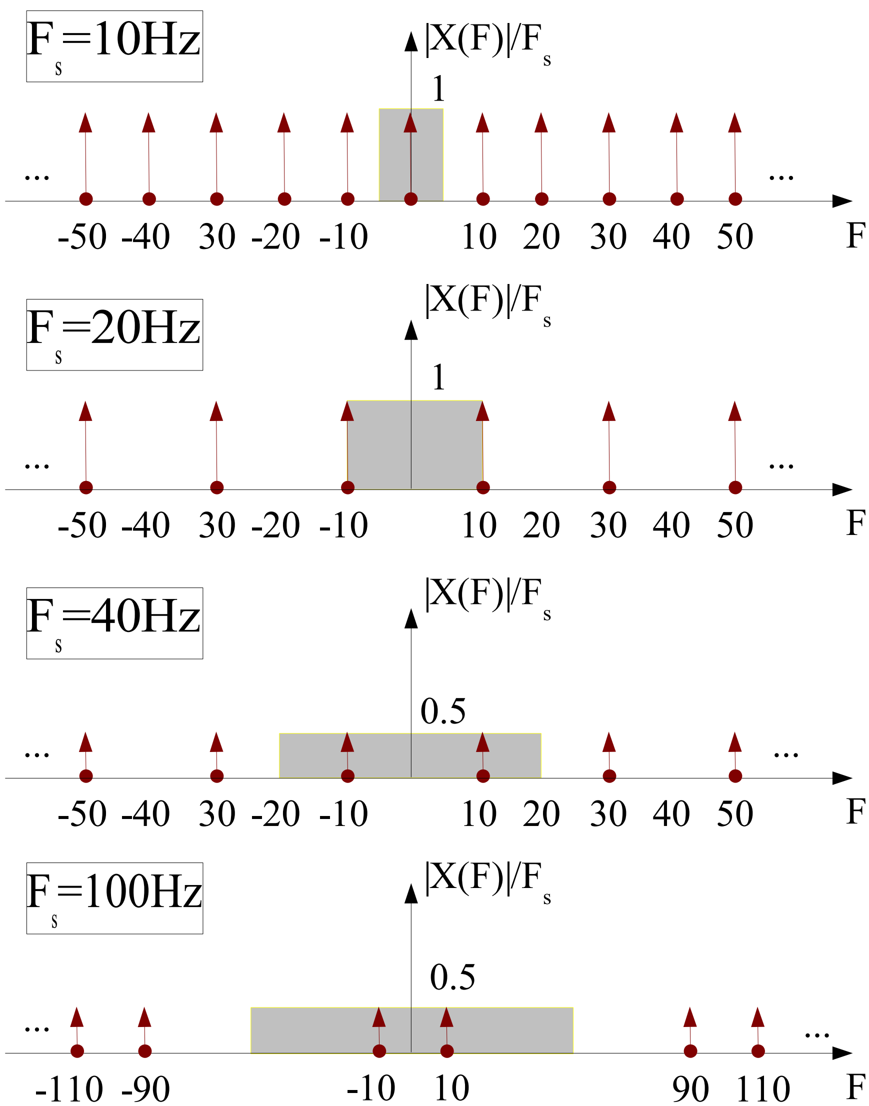


Figure 10.22: Spectrum of $X(F)$ for sampling frequencies $F_s = 10, 20, 40, 100$ Hz

for any value of Δ. *(Hint: consider the periodic nature of* $\tilde{F}(j\Omega)$ *and remember that a periodic function has a* Fourier series *expansion).*

(b) Give a qualitative description of the result.

(c) For $F(j\Omega)$ *as in Figure 10.23, sketch the resulting approximation* $\hat{F}(j\Omega)$ *for* $\Delta = 2\pi/\Omega_0$, $\Delta = \pi/\Omega_0$ *and* $\Delta = \pi/(100/\Omega_0)$.

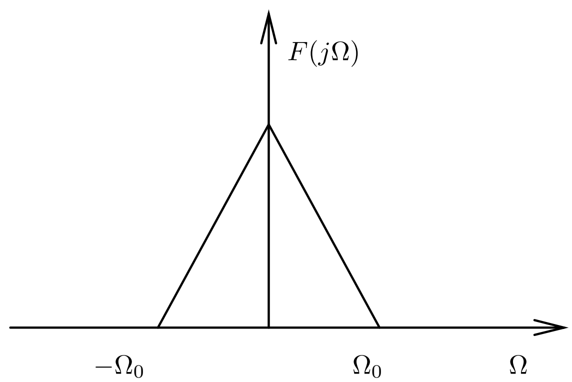


Figure 10.23: Fourier transform $F(j\Omega)$ in Problem 10.1.

Problem 10.2 *One of the standard ways of describing the sampling operation relies on the concept of "modulation by a pulse train". Choose a sampling interval* T_s *and define a continuous-time pulse train* $p(t)$ *as:*

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s).$$

The Fourier Transform of the pulse train is

$$P(j\Omega) = (2\pi/T_s) \sum_{k=-\infty}^{\infty} \delta(\Omega - k(2\pi/T_s))$$

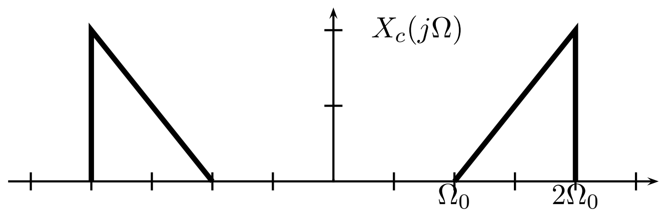
This is tricky to show, so just take the result as is. The "sampled" signal is simply the modulation of an arbitrary-continuous time signal $x(t)$ *by the pulse train:*

$$x_s(t) = p(t)\, x(t)$$

This sampled signal is still continuous but, by the properties of the delta function, it is nonzero only at multiples of T_s*; in a sense,* $x_s(t)$ *is a discrete-time signal brutally embedded in the continuous time world.*

Here's the question: derive the Fourier transform of $x_s(t)$ and show that if $x(t)$ is bandlimited to π/T_s then we can reconstruct $x(t)$ from $x_s(t)$.

Problem 10.3 *Consider a real, continuous-time signal $x_c(t)$ with the following spectrum:*



(a) *What is the bandwidth of the signal? What is the minimum sampling period in order to satisfy the sampling theorem?*

(b) *Take a sampling period $T_s = \pi/\Omega_0$; clearly, with this sampling period, there will be aliasing. Plot the DTFT of the discrete-time signal $x_a[n] = x_c(nT_s)$.*

(c) *Suggest a block diagram to reconstruct $x_c(t)$ from $x_a[n]$.*

(d) *With such a scheme available, we can therefore exploit aliasing to reduce the sampling frequency necessary to sample a bandpass signal. In general, what is the minimum sampling frequency to be able to reconstruct with the above strategy a real signal whose frequency support on the positive axis is $[\Omega_0, \Omega_1]$ (with the usual symmetry around zero, of course)?*

Appendix 10.A The Sinc Product Expansion Formula

The goal is to prove the product expansion

$$\frac{\sin(\pi t)}{\pi t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2}\right). \tag{10.49}$$

We will present two proofs; the first was proposed by Euler in 1748 and, while it certainly lacks rigor by modern standards, it has the irresistible charm of elegance and simplicity in that it relies only on basic algebra. The second proof is more rigorous, and is based on the theory of Fourier series for periodic functions; relying on Fourier theory, however, hides most of the convergence issues.

Euler's Proof Consider the N roots of unity for N odd. They will be $z = 1$ plus $N - 1$ complex conjugate roots of the form $z = e^{\pm j\omega_N k}$ for $k = 1, \ldots, (N-1)/2$ and $\omega_N = 2\pi/N$. If we group the complex conjugate roots pairwise we can factor the polynomial $z^N - 1$ as

$$z^N - 1 = (z-1) \prod_{k=1}^{(N-1)/2} (z^2 - 2z\cos(\omega_N k) + 1).$$

The above expression can be immediately generalized to

$$z^N - a^N = (z-a) \prod_{k=1}^{(N-1)/2} (z^2 - 2az\cos(\omega_N k) + a^2).$$

Now replace z and a in the above formula by $z = (1+x/N)$ and $a = (1-x/N)$; we obtain:

$$\begin{aligned}
\left(1+\frac{x}{N}\right)^N - \left(1-\frac{x}{N}\right)^N &= \frac{4x}{N} \prod_{k=1}^{(N-1)/2} \left((1-\cos(\omega_N k)) + \frac{x^2}{N^2}(1+\cos(\omega_N k))\right) \\
&= \frac{4x}{N} \prod_{k=1}^{(N-1)/2} (1-\cos(\omega_N k)) \left(1 + \frac{x^2}{N^2} \cdot \frac{1+\cos(\omega_N k)}{1-\cos(\omega_N k)}\right) \\
&= Ax \prod_{k=1}^{(N-1)/2} \left(1 + \frac{x^2\,(1+\cos(\omega_N k))}{N^2\,(1-\cos(\omega_N k))}\right)
\end{aligned}$$

where A is just the finite product $(4/N)\prod_{k=1}^{(N-1)/2}(1-\cos(\omega_N k))$. The value A is also the coefficient for the degree-one term x in the right-hand side, and it can be easily seen from the expansion of the left hand-side that $A = 2$ for all N; actually, this is an application of Pascal's triangle, and it was proven by Pascal in the general case in 1654. As N grows large we have that

$$\left(1 \pm \frac{x}{N}\right)^N \approx e^{\pm x};$$

at the same time, if N is large, then $\omega_N = 2\pi/N$ is small and, for small values of the angle, the cosine can be approximated as

$$\cos(\omega) \approx 1 - \omega^2/2$$

so that the denominator in the general product term can in turn be approximated as:

$$N^2(1-\cos((2\pi/N)k) \approx N^2 \cdot \frac{4k^2\pi^2}{2N^2} = 2k^2\pi^2.$$

By the same token, for large N, the numerator can be approximated as $1+\cos((2\pi/n)k) \approx 2$ and therefore the above expansion becomes (by bringing $A = 2$ over to the left-hand side):

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\ldots$$

Finally, we replace x by $j\pi t$ to obtain

$$\frac{\sin(\pi t)}{\pi t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2}\right).$$

Rigorous Proof Consider the Fourier series expansion of the *even* function $f(x) = \cos(\tau x)$ periodized over the interval $[-\pi, \pi]$. We have

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

with

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\tau x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}[\cos((\tau+n)x) + \cos((\tau-n)x)]dx \\ &= \frac{1}{\pi}\left[\frac{\sin((\tau+n)\pi)}{\tau+n} + \frac{\sin((\tau-n)\pi)}{\tau-n}\right] \\ &= \frac{2\sin(\tau\pi)}{\pi}\,\frac{(-1)^n\tau}{\tau^2-n^2} \end{aligned}$$

so that

$$\cos(\tau x) = \frac{2\tau\sin(\tau\pi)}{\pi}\left(\frac{1}{2\tau^2} - \frac{\cos(x)}{\tau^2-1} + \frac{\cos(2x)}{\tau^2-2^2} - \frac{\cos(3x)}{\tau^2-3^2} + \ldots\right)$$

In particular, for $x = \pi$ we have

$$\cot(\pi\tau) = \frac{2\tau}{\pi}\left(\frac{1}{2\tau^2} + \frac{1}{\tau^2-1} + \frac{1}{\tau^2-2^2} + \frac{1}{\tau^2-3^2} + \ldots\right)$$

which we can rewrite as

$$\pi\left(\cot(\pi\tau) - \frac{1}{\pi\tau}\right) = \sum_{n=1}^{\infty} \frac{-2\tau}{n^2-\tau^2}$$

If we now integrate between 0 and t both sides of the equation we have:

$$\int_0^t \left(\cot(\pi\tau) - \frac{1}{\pi\tau}\right) d\pi\tau = \ln \frac{\sin(\pi\tau)}{\pi\tau}\bigg|_0^t = \ln\left[\frac{\sin(\pi t)}{\pi t}\right]$$

and

$$\int_0^t \sum_{n=1}^{\infty} \frac{-2\tau}{n^2-\tau^2}\, d\tau = \sum_{n=1}^{\infty} \ln\left(1-\frac{t^2}{n^2}\right) = \ln\left[\prod_{n=1}^{\infty}\left(1-\frac{t^2}{n^2}\right)\right]$$

from which, finally,

$$\frac{\sin(\pi t)}{\pi t} = \prod_{n=1}^{\infty}\left(1-\frac{t^2}{n^2}\right).$$