Signal Processing for Communications
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Instructor: Prof. Suhas Diggavi
LICOS, EPFL

based on lecture notes by Suhas Diggavi (LICOS), Paolo Prandoni and Martin Vetterli (LCAV), EPFL

Contents

Chapter 0

Mathematical Prerequisites

0.1 Complex Numbers

Complex number is a number of the form $a+bi$, where a and b are real numbers, and i is the imaginary unit, with the property $i^2 = -1$. The real number a is called the real part of the complex number, and the real number b is the imaginary part. When the imaginary part b is 0, the complex number is just the real number a.

For example, $3+2i$ is a complex number, with real part 3 and imaginary part 2. If $z = a+bi$, the real part (a) is denoted $Re(z)$, and the imaginary part (b) is denoted $Im(z)$.

Complex numbers can be added, subtracted, multiplied, and divided like real numbers, but they have additional elegant properties. For example, real numbers alone do not provide a solution for every polynomial algebraic equation with real coefficients, while complex numbers do (the fundamental theorem of algebra).

In some fields (in particular, electrical engineering and electronics, where i is a symbol for current), complex numbers are written as $a+bj$.

0.1.1 Operations on Complex Numbers

The set of all complex numbers is usually denoted as $\mathbb{C}$. The additions, substractions and multiplications of complex numbers follow the associative, commutative and distributive laws of algebra. Combining the latter properties with the equation $i^2 = -1$, it is easy to see that:

$$(a+bi)+(c+di) = (a+c)+(b+d)i \tag{0.1}$$

$$(a+bi)-(c+di) = (a-c)+(b-d)i \tag{0.2}$$

$$(a+bi)(c+di) = ac+bci+adi+bdi^2 = (ac-bd)+(bc+ad)i \tag{0.3}$$

0.1.2 The Complex Number Field

Formally, the complex numbers can be defined as ordered pairs of real numbers (a, b) together with the operations:

$$(a, b) + (c, d) = (a + c, b + d) \tag{0.4}$$
$$(a, b) \cdot (c, d) = (ac - bd, bc + ad) \tag{0.5}$$

So defined, the complex numbers form a field, the complex number field, denoted by $\mathbb{C}$.

Since a complex number $a + bi$ is uniquely specified by an ordered pair (a, b) of real numbers, the complex numbers are in one-to-one correspondence with points on a plane, called the *complex plane*.

We identify the real number a with the complex number $(a, 0)$, and in this way the field of real numbers $\mathbb{R}$ becomes a subfield of $\mathbb{C}$. The imaginary unit i is the complex number $(0, 1)$.

In $\mathbb{C}$, we have:

- additive identity (”zero”): $(0, 0)$
- multiplicative identity (”one”): $(1, 0)$
- additive inverse of $(a, b) : (-a, -b)$
- multiplicative inverse (reciprocal) of non-zero (a, b): $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

0.1.3 The Complex Plane

A complex number $z = a + bi$ can be viewed as a point or a position vector on a two-dimensional Cartesian coordinate system called the complex plane. The Cartesian coordinates of the complex number are the real part a and the imaginary part b, while the polar coordinates are $r = |z|$, called the absolute value or modulus, and $\phi = arg(z)$, called the complex argument of z (mod-arg form). Together with Euler’s formula we have (see figure 0.1)

$$z = a + bi = r(\cos\varphi + i\sin\varphi) = re^{i\varphi} \tag{0.6}$$

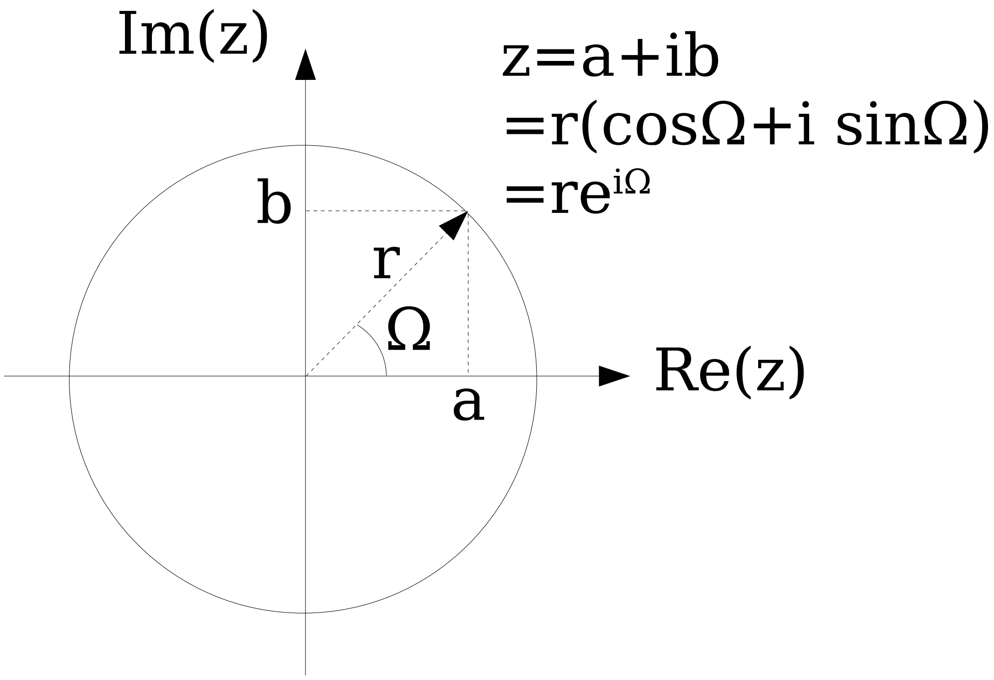


Figure 0.1: Complex plane representation

It is easy to see that:

$$r = |z| = \sqrt{a^2 + b^2} \tag{0.7}$$

$$tan\varphi = \frac{b}{a} \tag{0.8}$$

$$cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \tag{0.9}$$

$$sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \tag{0.10}$$

By simple trigonometric identities, we see that

$$r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1 + \varphi_2)} \tag{0.11}$$

and that

$$\frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}. \tag{0.12}$$

This gives you an easy way to calculate the powers and the roots of complex numbers. Pay attention to the fact that $re^{i\varphi} = re^{i(\varphi+2k\pi)}$, $k \in \mathbb{Z}$, so $(re^{i\varphi})^{1/N} = r^{1/N}e^{i(\varphi+2k\pi)/N}$,$k = 0, 1, ..., N-1$.

Now the addition of two complex numbers is just the vector addition of two vectors, and the multiplication with a fixed complex number can be seen as a simultaneous rotation and stretching.

Multiplication with i corresponds to a counter clockwise rotation by 90 degrees ($\pi/2$ radians). The geometric content of the equation $i^2 = -1$ is that a sequence of two 90 degree rotations results in a 180 degree (π radians) rotation. Even the fact (-1) (-1) = +1 from arithmetic can be understood geometrically as the combination of two 180 degree turns.

0.1.4 Absolute Value, Conjugation and Distance

One can check readily that the absolute value has three important properties:

$$|z| = 0 \ if \ and \ only \ if \ z = 0 \tag{0.13}$$

$$|z + w| \leq |z| + |w| \ (triangle \ inequality) \tag{0.14}$$

$$|zw| = |z| \cdot |w| \tag{0.15}$$

for all complex numbers z and w. It then follows, for example, that $|1| = 1$ and $|z/w| = |z|/|w|$. By defining the distance function $d(z, w) = |z - w|$ we turn the complex numbers into a metric space and we can therefore talk about limits and continuity. The addition, subtraction, multiplication and division of complex numbers are then continuous operations. Unless anything else is said, this is always the metric being used on the complex numbers.

The complex conjugate of the complex number $z = a+ib$ is defined to be $a-ib$, written as $\bar{z}$ or z^*. $\bar{z}$ is the "reflection" of z about the real axis. The following can be checked:

$$\overline{z + w} = \bar{z} + \bar{w} \tag{0.16}$$

$$\overline{zw} = \bar{z}\bar{w} \tag{0.17}$$

$$\overline{(z/w)} = \bar{z}/\bar{w} \tag{0.18}$$

$$\bar{\bar{z}} = z \tag{0.19}$$

$$\bar{z} = z \ if \ and \ only \ if \ z \ is \ real \tag{0.20}$$

$$|z| = |\bar{z}| \tag{0.21}$$

$$|z|^2 = z\bar{z} \tag{0.22}$$

$$z^{-1} = \bar{z}|z|^{-2} \ if \ z \ is \ non - zero. \tag{0.23}$$

The latter formula is the method of choice to compute the inverse of a complex number if it is given in rectangular coordinates.

That conjugation commutes with all the algebraic operations (and many functions; e.g. $\sin \bar{z} = \overline{\sin z}$) is rooted in the ambiguity in choice of i (-1 has two square roots).

0.2 Summations

Let f be a function whose domain includes the integers from n through m. We define

$$\sum_{i=n}^{m} f(i) = f(n) + f(n+1) + ... + f(m) \tag{0.24}$$

We call i the index of summation,n is the lower limit of summation, and m is the upper limit of summation. One can show that:

$$\sum_{k=1}^{n} c = c + c + ... + c = cn \tag{0.25}$$

$$\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2} \tag{0.26}$$

$$\sum_{k=1}^{n} k^2 = 1 + 4 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{0.27}$$

$$\tag{0.28}$$

Another well-known result is the following:

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + ... + r^n = \begin{cases} \frac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ n+1 & \text{else} \end{cases} \tag{0.29}$$

Note that when $r < 1$:

$$\lim_{n \to +\infty} S_n = \frac{1}{1-r} \tag{0.30}$$

Additionally, be very cautious when taking squares of summations:

$$\left[\sum_{i=n}^{m} f(i)\right]^2 = \left[\sum_{l=n}^{m} f(l)\right] \left[\sum_{k=n}^{m} f(k)\right] = \sum_{l=n}^{m} \sum_{k=n}^{m} f(l) f(k) \tag{0.31}$$

Finally, let's $S_N = \sum_{n=1}^{N} a_n$ and $S = \lim_{N \to +\infty} S_N$. If the sequence of partial sums is divergent (i.e. either the limit does not exist or is infinite) then we call the series divergent.

if $|S| = c < \infty$, we call the series convergent and we call S the sum or value of the series. The Cauchy convergence criterion states that a series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is a Cauchy sequence. This means that for every $\varepsilon > 0$, there is a positive integer N such that for all $n \geq m \geq N$ we have:

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \tag{0.32}$$

which is equivalent to

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{k=n}^{n+m} a_k = 0 \tag{0.33}$$

0.3 Integration

Besides being comfortable with the basic properties of integrals and methods for integration (e.g. substitution, integration by parts) it is important to know the definition and basic properties of the convolution integral.

The convolution between two functions f and g, both with domain $\mathbb{R}$, is itself a function, let's call it h, and is defined by

$$h(x) := f(x) * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

The following properties are easy to show:

- $f(x) * g(x) = g(x) * f(x)$.
- $f(x) * (g(x) * h(x)) = (f(x) * g(x)) * h(x)$.
- $f(x) * (\alpha \cdot g(x) + \beta \cdot h(x)) = \alpha \cdot f(x) * g(x) + \beta \cdot f(x) * h(x)$.

0.4 Linear Algebra

0.4.1 Matrices

Let A be a matrix with n rows and m colums of complex entries. That is, we have

$$A := \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & \dots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \dots & A_{n,m} \end{bmatrix},$$

$A_{i,j} \in \mathbb{C}$.

One of the basic operations on A is taking the *transpose*, denoted by A^T and defined as $\left(A^T\right)_{i,j} = A_{j,i}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. More explicitely we have

$$A^T = \begin{bmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,m} & A_{2,m} & \dots & A_{m,n} \end{bmatrix}.$$

The *conjugate transpose* of A, denoted by A^*, is defined as $(A^*)_{i,j} = A^*_{j,i}$. Besides taking the transpose of A we take the complex conjugate of each element. Note that A^* is also known as the Hermitian of A.

Based on the above operations we define *symmetric* and *Hermitian* matrices. A real matrix A is symmetric if $A^T = A$, a (complex) matrix is Hermitian if $A^* = A$.

The matrix A can be right multiplied with a m by p matrix, say B, resulting in a n by p matrix. Remember that matrix multiplication is defined by $(AB)_{i,j} = \sum_{k=1}^{m} A_{i,k}B_{k,j}$. Note that **matrix multiplication is not commutative**, i.e. $AB \neq BA$ (assuming $n = m$).

0.4.2 Vectors

Let c and d be length n resp. m vectors, i.e. $c = [c_1, c_2, \ldots, c_n]$ and $d = [d_1, d_2, \ldots, d_m]$. We will usually assume that vectors are column vectors. This allows us to right multiply our matrix A with vector d. The result Ad is a length n vector. Similarly $b^T A$ gives a length m row vector.

The *inner product* between two n length vectors a and b is defined by

$$\langle a, b\rangle \ := \sum_{i=1}^{n} a_i b_i = a^T b.$$

Note that matrix multiplication is nothing more than taking inner products between rows and columns of the two matrices. The most common way to define the *norm* of a vector is through the inner product. This gives that the norm of x, $\|x\|_2$ is defined as

$$\|x\|_2 = \langle x, x\rangle^{1/2}.$$

A very useful relation is the *Cauchy-Schwartz inequality*, which states that

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2.$$

0.4.3 Determinants

One of the most used properties of a matrix is it's determinant. The determinant of a 2 by 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\det(A) = ad - bc.$$

In general for a square n by n matrix A we have, for any row $i = 1, \ldots, n$

$$\det(A) = \sum_{j=1}^{n} A_{i,j}(-1)^{i+j} \det\left(A^{\setminus(i,j)}\right),$$

where $A^{\setminus(i,j)}$ is the resulting matrix after removing row i and column j from matrix A. We can also expand along any column $j = 1, \ldots, m$, which gives us

$$\det(A) = \sum_{i=1}^{n} A_{i,j}(-1)^{i+j} \det\left(A^{\setminus(i,j)}\right).$$

An important result to keep in mind is that

a matrix is invertible if and only if it's determinant is not equal to zero.

Finally, we note the following basic relations:

- $\det(AB) = \det(A)\det(B)$.
- $\det\left(A^{-1}\right) = \det\left(A\right)^{-1}$.
- $\det(A^*) = \det(A)^*$.

0.4.4 Eigenvalues and Eigenvectors

The *eigenvalues* of a matrix A are the solutions for λ in the equation

$$\det(A - \lambda I) = 0,$$

which is called the *characteristic equation*. Given that $\tilde{\lambda}$ is an eigenvalue of A, we call the vector x for which

$$Ax = \tilde{\lambda}x.$$

the *eigenvector* corresponding to $\tilde{\lambda}$.

One can verify that the eigenvalues of A and A^* are the same. The eigenvectors for a matrix and it's conjugate transpose are however different.

0.5 Problems

Problem 0.1 *1. Let* $s[n] := \frac{1}{2^n} + j\frac{1}{3^n}$. *Compute* $\sum_{n=1}^{\infty} s[n]$.

2. *Same question with* $s[n] := (\frac{j}{3})^n$.

3. *Characterize the set of complex numbers satisfying* $z^* = z^{-1}$.

4. *Find 3 complex numbers* $\{z_0, z_1, z_2\}$ *which satisfy* $z_i^3 = 1$, $i = 1, 2, 3$.

5. *What is the following infinite product* $\prod_{n=1}^{\infty} e^{j\pi/2^n}$ *?*

Problem 0.2 (Geometric Series) *Consider the sequence* $x[n] = a \cdot r^n$ *for some real* r. *Let* $S[n] = \sum_{k=0}^{n} x[k]$.

(a) *The goal is to find a closed expression for* $S[n]$.

- *Compare the two sequences* $S[n]$ *and* $S[n+1]$.
- *Multiply each term in* $S[n]$ *by* r *and obtain* $\tilde{S}[n] = rS[n]$. *Compare* $\tilde{S}[n]$ *to* $S[n+1]$.
- *We have obtained a system of two equations in the unknowns* $S[n]$ *and* $S[n+1]$. *Solve this system and find the expression of* $S[n]$ *in terms of* a, r *and* n.

(b) *Let* $|r| < 1$ *and find* $S[n]$ *when* n *goes to infinity,* i.e., $S = \sum_{k=0}^{\infty} x[k]$.

(c) *Find an expression for the summation* $\sum_{k=n+1}^{m} x[k]$.

(d) *Apply the obtained formula and compute* $\sum_{k=0}^{\infty} t[k]$, *where* $t[k] = \frac{1}{3^k} + (\frac{1}{2j})^k$.

(e) *How can we use this formula to compute* $\Pi_{n=1}^{\infty} e^{j\pi/2^n}$ *?*

Problem 0.3 (Complex Numbers) (a) *Find all the roots of* $x^3 + 2x^2 + 2x + 1 = 0$. *What is the summation of the roots?*
Hint: *Try to check small integers and find one of the roots and then solve the remaining degree 2 polynomial*

(b) *Compute* j^j, *where* $j = \sqrt{-1}$.

(c) *Consider the polar representation of the complex numbers and find all which satisfy* $\arg(z) = |z|$. *(Note that* $0 \leq \arg(z) \leq 2\pi$*)*

(d) *Characterize the set of complex numbers satisfying* $z^* = z^{-1}$.

Problem 0.4 (Linear Algebra) *(a) Compute the determinant for the following matrix.*

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

(b) Consider the matrices

$$B = \begin{bmatrix} j & -1 & 4 \\ 0 & 2-3j & 1 \\ -1 & 2j & 0 \\ 3 & 0 & 4-j \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & j & 1 \\ 1-5j & 1 & 4j & 2+2j \\ 1 & 3-j & 0 & -7 \end{bmatrix}.$$

Which of the following operations are well-defined (Note that you do NOT have to compute)? $C+B$, $C \cdot A^{-1}$, $B \cdot C$, $A - C$, $B + B^T$, $A + A^T$, $C^{-1} \cdot B^{-1}$, $C^* + B$,

(c) Let $x = [1,\ 2j,\ 1+j,\ 0]$. *Compute* Ax^T *and* xB.

(d) Compute the determinant of $D = xx^T$ *and* $E = x^T x$.

Chapter 1

What Is Signal Processing ?

1.1 Introduction

As implied by the name, signal processing deals with signals on the one hand, and operations on signals on the other hand. That is, the "black box" view of signal processing is as shown in this scheme:

The purpose of these notes is to understand what a signal is, how it relates to the world around us, and how we can manipulate it. That is, we want to understand what to put inside the "black box" of the previous figure in order to be able to treat relevant signals from the real world, and produce output signals as required by applications, in particular in the context of communication systems.

As stated above, the realm of signal processing might seem too vague, or too all-encompassing. Any physical quantity evolving over time or space qualifies as a signal, and any possible computation performed on such a signal is a signal processing operation. That is, in many areas of applied sciences and engineering, people run what are essentially "signal processing" algorithms, not always knowingly. Or, to paraphrase Molière's Monsieur Jourdain, many people "font du traitement du signal sans le savoir". Our aim will not be to claim as large a field as possible, but to clearly specify that there are many other possible applications beyond those which we will study in detail; in our case, the focus will be mostly on telecommunications and related fields in electrical engineering and computer sciences. Whenever relevant, we will also try to illustrate the overlapping between signal processing and other scientific disciplines such as acoustics, statistics, geophysics, applied and computational harmonic analysis, and so on.

The outline of this introductory chapter is as follows. First, we will showcase a "gallery of signals", from the floods of the Nile to the stock market, in order to point out the common traits and the differences between various signals. Next we will describe a series of prototypical "black box" systems, ranging from the very simple to the very complicated, all of which perform some meaningful signal processing task. We will then discuss the idea of an "underlying model" for a signal, which can be put to advantage in the design of signal processing algorithms. We will then sketch a very brief history of modern signal processing, highlighting the main achievements and their impact. Finally we will conclude the chapter with an overview of the structure of these notes. But before moving on, we need to introduce at some initial, elementary technical concepts, which will then be refined along the way.

1.2 Elementary Concepts

Let $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the set of integer, real, and complex numbers respectively. Most real world signals can be modeled as real or complex functions of one or more real arguments. Signals of this type are called *continuous-time signals* and will be indicated by the familiar lowercase notation used for real function, e.g. $f(t)$, $x(t)$, and so on. Signals which are functions of a single real argument are called one-dimensional signal and in this case the real argument usually represents time, while the signal is the evolution of a given quantity (often an electrical signal) over time. Functions of two arguments are called two-dimensional signals and the two arguments usually represent a coordinate pair in the plane; this is for instance the case of a signal representing an image. For most continuous-time signals, considered as functions on the real line, the Fourier transform (as defined in the standard calculus course) is a well defined operator; we will indicate the Fourier transform of a signal $f(t)$ as $F(j\Omega)$.

A discrete-time signal is a sequence of numbers (real or complex) indexed by one or more integer arguments. Again, a lower case letter will represent a given sequence but, in order to make explicit the discrete nature of the argument, the latter is enclosed in square brackets: $f[n]$, $x[n]$, and so on. This somewhat unusual notation is actually quite standard in the signal processing literature. Now, discrete-time signals are very often obtained from "sampling" a continuous-time signal. By sampling, we mean taking the value of a signal at regular intervals, or integer multiples of a sampling period T. This process will be studied in detail in Chapter 10, but we shall note right away that sampling a continuous-time signal $f(t)$ leads to a discrete-time signal $f[n]$ (please note the round parenthesis and the square brackets):

$$f[n] = f(nT) \quad n \in \mathbb{Z} \tag{1.1}$$

where T is a real number greater than 0. A key question is of course whether we can recover $f(t)$ from its samples $f[n]$, a question we will settle in the aforementioned Chapter 10. A Fourier representation indeed exists for most discrete-time signals as well, and its study will be the subject of Chapter 3. In general, the Fourier transform of a discrete-time sequence $f[n]$ will be indicated by $F(e^{j\omega})$.

Sometimes, one considers a *finite-length* signal, with indexes $0 \ldots N-1$ where N denotes the length. It is then natural to consider such a signal as a vector in $\mathbb{R}^N$ or $\mathbb{C}^N$, the N-dimensional real and complex Euclidean spaces. In such a case, we will use linear algebra notation, that is, vectors will be denoted by lower case bold letters, e.g. $\mathbf{f}, \mathbf{x}$, etc. Linear transforms are given by matrices, denoted by bold upper case matrices, e.g. $\mathbf{M}$. For example, a finite-dimensional version for the Fourier transform, called the discrete Fourier transform (DFT) and discussed in Section 3.3, will map an N-dimensional "signal" $\mathbf{x}$ into its N-dimensional transform $\mathbf{y}$ with a special matrix $\mathbf{W}$ as:

$$\mathbf{y} = \mathbf{W} \cdot \mathbf{x} \tag{1.2}$$

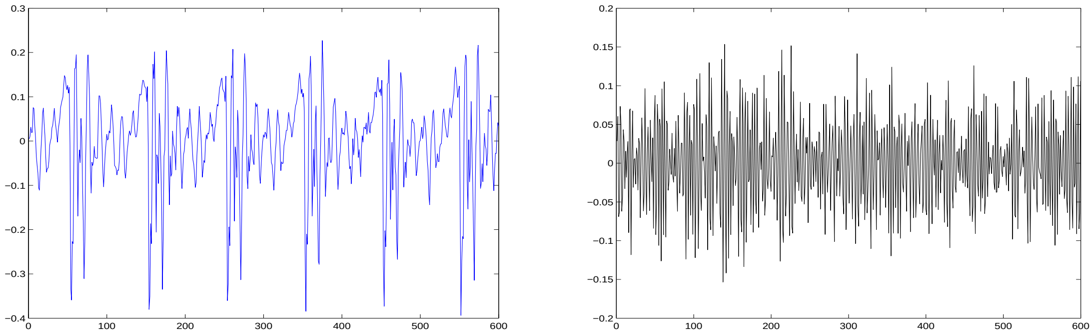
This concludes our elementary notations, where we have introduced three types of objects central to signal processing; these are summarized in the Table 1.1, which will be our rudimentary road map for the rest of the chapter.

Signal Type	**Time-domain Notation**	**Frequency-domain Notation**
Continuous-time	$x(t), \quad t \in \mathbb{R}$	$X(j\Omega)$
Discrete-time	$x[n], \quad n \in \mathbb{N}$	$X(e^{j\omega})$
Finite-length	$\mathbf{x}, \quad \mathbf{x} \in \mathbb{C}^N$	$\mathbf{X} = \mathbf{W}\mathbf{x}$

Table 1.1: Fundamental signal types.

1.3 Examples of Signals

We will now present a series of signals drawn from our everyday experience. Most of these are signals which are "processed" by our senses (hearing, vision). Others are more



(a) (b)

Figure 1.1: Speech signals. (a) Voiced sound, corresponding to the sound "a", (b) Unvoiced sound corresponding to "shh...".

abstract and represent our analysis of real-world phenomena.

1.3.1 Speech

For a human being, the most natural signal is depicted in Figure 1.1. It is a speech signal in two of its typical forms, i.e. *voiced* speech in part (a) and *unvoiced* speech in part (b). It is fairly obvious from the picture that voiced sounds look almost periodic, while unvoiced speech possesses a noise-like character. One fact is immediately obvious from Figure 1.1: most of us would not recognize the vowel "a" or the sound "sh" from the plot of the signal, yet all of us will instantly recognize the sounds when played to our ears. Welcome to signal processing! A task which is trivial for the human ear and its attached processor (the brain) becomes a very hard problem for a computer. In particular, computer recognition of continuous speech (as opposed to isolated words like digits for example) from an unknown speaker (that is, without training tuned for a particular speaker) is still an open problem, even though a child can master it easily. Speech has fascinated signal processing researchers for decades, and due to its obvious economical importance, vast amounts of work have gone into "understanding" the speech signal; the two main lines of research involve studying speech production, i.e. how humans produce speech signals, and speech recognition, i.e. how humans perceive and analyze speech.

Speech can clearly be modeled as a continuous-time signal, corresponding to sound waves produced by the vocal cords and filtered by the trachea and mouth. Yet, most of speech processing (except in "old" analog telephony) is done in discrete-time, typically at a sampling rate of 8kHz, or a sampling interval of $T = 0.125$ ms. As we will see, such sampling does not impair the signal very much, and is thus standard in "digital" telephony.

1.3.2 Music

Next to speech, musical sounds are the most ancient signals produced by humans. Figure 1.2 shows several examples of signals produced by musical instruments, ranging from the "simplest" to the most "complex". As quite obvious from looking at the picture (and even more so by listening to the sound!), there is a wide variety of musical sounds, from the simple, sinusoidal flute to the polyphonic complexity of a full orchestra. Music through the ages has been about synthesizing interesting sounds, and signal processing has added many new "instruments" to the toolbox in the last decades. From vocoders (to help aspiring singers stay in tune...) to full blown synthesizers able to imitate almost any instrument to near perfection, examples abound where sophisticated signal processing techniques create "new sounds". Most are based on prior fine analysis of actual instruments, in order to best imitate them, while others are completely "artificial". Depending on one's musical taste, some of the achievements have been a mixed blessing for music. A few examples are given in Figure 1.3.

Besides producing music, signal processing is also involved in the more mundane task of recording, storage, transmission, etc. Again, while the signal is a sound wave, a discrete-time version with sampling at 44 KHz is most often used, with little degradation.

1.3.3 Other One-Dimensional Signals

So far, we have looked at the two most natural signals, namely speech and music. Many other examples are possible, and we will pick a few outstanding examples. Figure 1.4 shows two key signals for humans, namely the electro-cardiogram, and the electro-encephalogram. These indicate heart and brain activity, respectively. Such signals have been studied by doctors for decades in order to monitor illness, or predict future problems. Continuous monitoring by means of automatic analysis is becoming a reality as processing has become sophisticated and reliability has increased (e.g. automatic defibrillation devices for heart patients).

Mother nature is a great producer of interesting signals, from seismic activity to solar spots. A famous example is what is considered to be on many accounts the *oldest discrete-time signal on record*. It is actually a time-series, with a sampling interval of one year which, amongst other things, indicates the height of the floods of the Nile in ancient Egypt from 2925 BC to 2325 BC. The floods of the Nile were studied in modern times by Hurst in order to spot any regularity. Instead, he found the "Hurst parameter", a fractal measure of

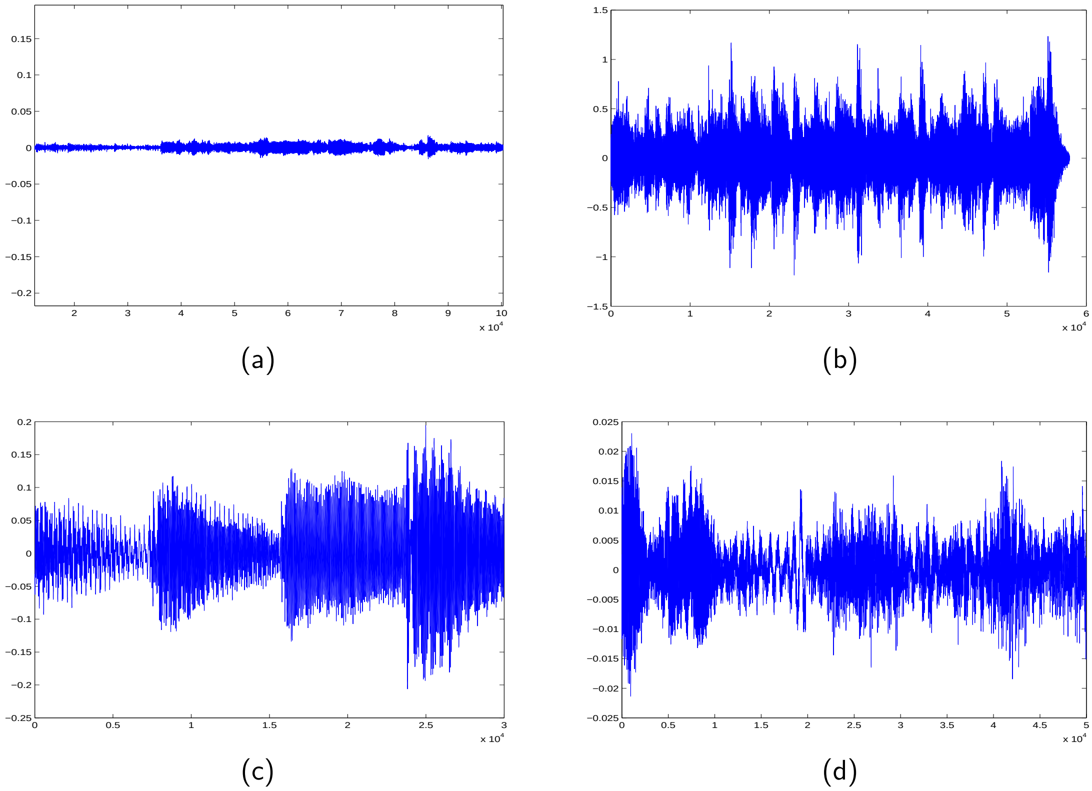


Figure 1.2: Examples of musical sounds. (a) Flute (beginning of Ravel's Bolero), (b) Full orchestra (ending of Ravel's Bolero), (c) Piano (from Bach's Goldberg Variations), (d) String quartet (from Schubert's Op.29).

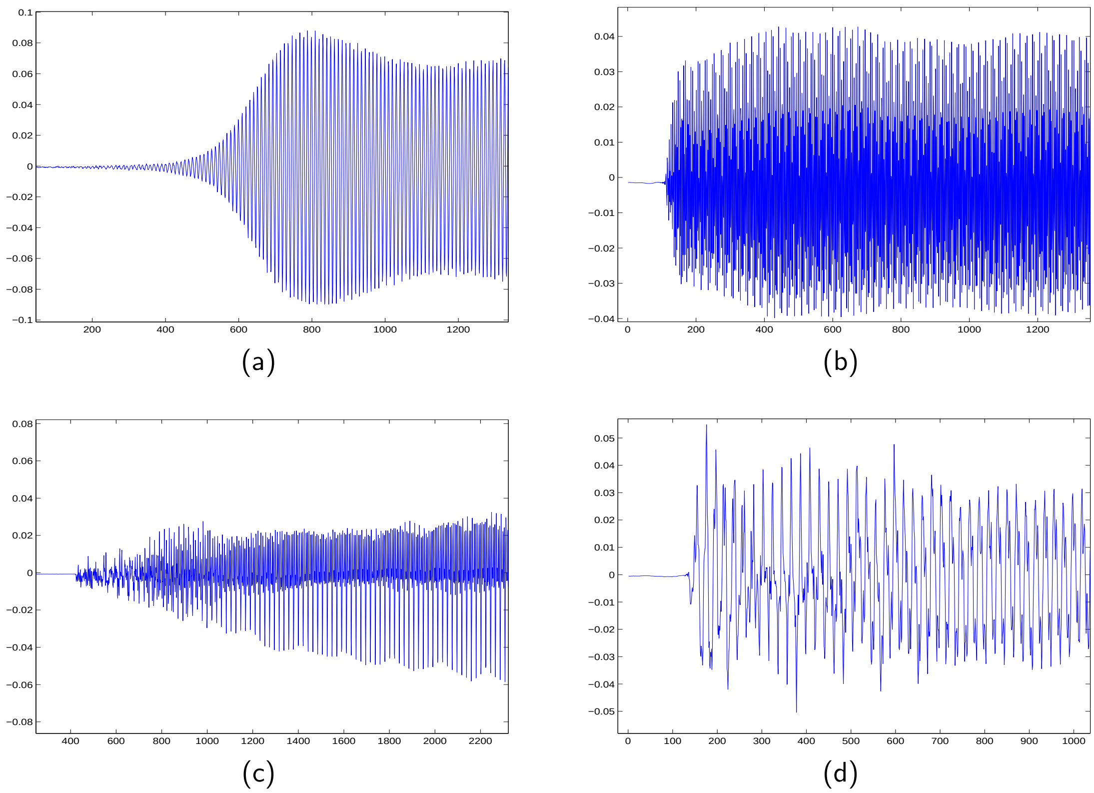
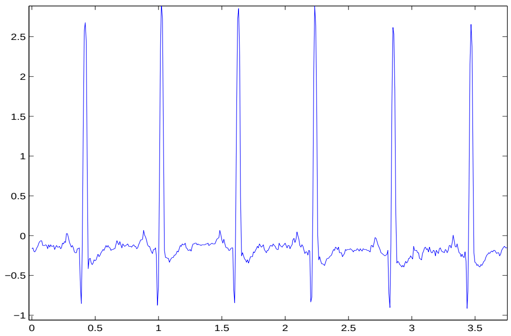
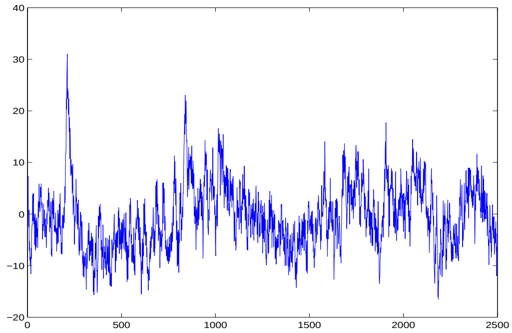


Figure 1.3: Examples of musical sounds from synthetic instruments. (a) Flute (with additive synthesis), (b) Trumpet (with FM modulation), (c) Violin (with physical modeling), (d) Piano (with digital waveguide).



(a)



(b)

Figure 1.4: Medical signals of vital importance.
(a) Electro-cardiogram (ECG), (b) Electro-encephalogram (EEG).

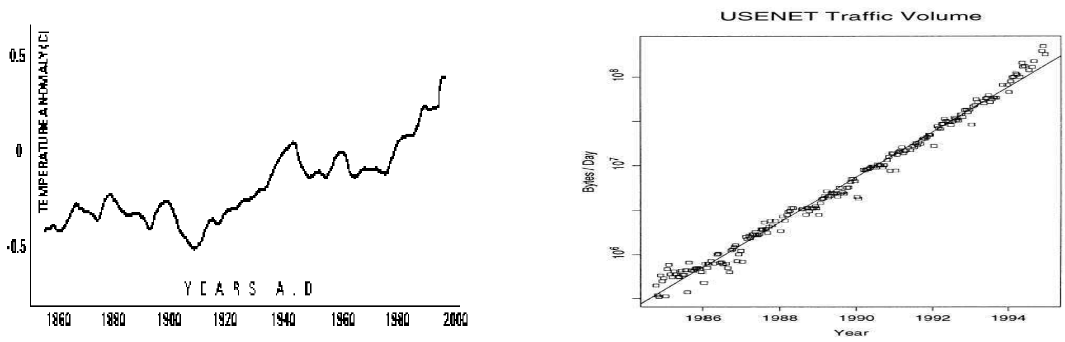
long-range dependence that is used today in the analysis of internet traffic; see Figure 1.5 for examples of both.

A famous time series that has obsessed many people is the stock market index, which is known for its trends but also its unpredictability due to abrupt changes (e.g. 1929...). The million dollar question is: What is the market value tomorrow? Yet another view of human activity is at the center of an intense scientific and political controversy. The signal at the heart of the debate is fairly simple, since it is the measure of the temperature on planet earth over the last hundred years or so. But there is much more at stake than in the stock market question: is there a global warming phenomenon, that could wipe out civilization as we know it? You can try your guess on Figure 1.6!

1.3.4 Images

So far, all signals we considered were functions of a single variable, typically time. If we look at signals of two variables, say $f(x, y)$ or $f[n, m]$, we get in the realm of image processing. Of course, long before the advent of photography and image processing, humans had “projected” images of the real world onto a two-dimensional surface, from Lascaux to Giza and more. Yet the modern age started with the invention of photography in the 19th century, which soon produced “scientific” images of the world around us.

A few particularities should be noted about images. Typically, images are of finite extent, whereas many one-dimensional signals had infinite length (ignoring the big-bang



(a) (b)

Figure 1.5: Long range dependent time-series. (a) The floods of the Nile in modern times, (b) Internet traffic.

for a moment, and hoping the best for the future...). Images are either black and white ($f(x, y)$ is real and positive) or color ($f(x, y)$ is a vector function depending on a color space). Finally, until recently, images were continuous in the space dimension, whereas now, many images are "digital" right from the start. An image in digital form is typically an array of 512 by 512 or 1024 by 1024 picture elements (pixels). It is to be noted that a high quality photograph on chemical film is of much higher quality, even though the gap is constantly narrowing.

1.3.5 Other Types of Images

Besides "classic" photographs, many other types of images are possible. For example, in the medical field, X-ray pictures and ultrasound images are very common, examples of which are shown in Figure 1.7. Clearly, such images have very different characteristics from "natural" images, a fact that obviously will influence the signal processing techniques used on such data.

1.3.6 Higher-Dimensional Signals

Stepping up from two to three dimensions, the dominant signal is certain video and film, that is "moving pictures". An important point to notice right away is that in such three-dimensional data, the time dimension is always discrete, e.g. 24 frames/sec in movies, 50 or 60 fields/sec in television. This is shown schematically in Figure 1.8.

Figure 1.6: A vital signal about the health of planet earth: temperature evolution from 1830 to 2000.

Moving pictures have fascinated viewers ever since their invention late in the 19th century. Because of the sampling in the time dimension, they are also one of the first examples of a sampled data system. This sampling can also cause some artifacts that are well known to fans of “western” movies (among others). It is the famous backward turning wagon-wheel effect, shown schematically in Figure 1.9. This is an example of aliasing, a phenomenon typical of sampling. The reason for the “visual illusion” of the wheel turning backward is that there are several possible continuous time events that map to the same sampled sequence, and the human eye will pick out the most likely. In the example above, let’s say that the wheel turns clockwise by $3\pi/4$ between each sampling instant. The sampled version is as in Figure 1.9. However, because of symmetry, the

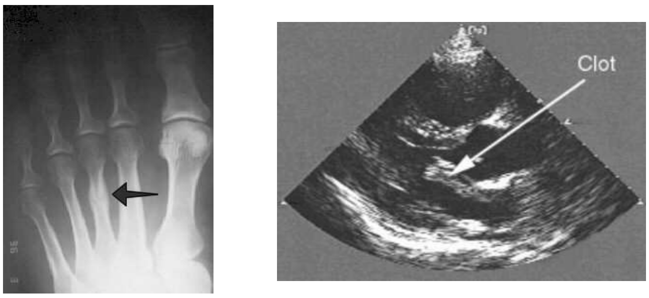


Figure 1.7: Medical images. (a) X-ray image (details), (b) Ultrasound image (details).

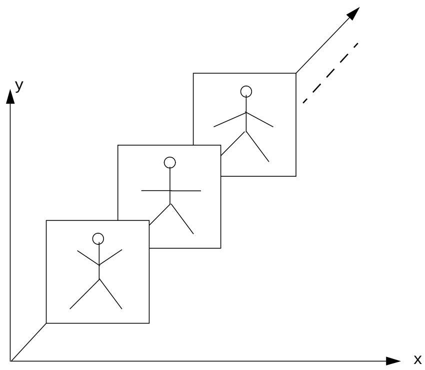


Figure 1.8: Three dimensional signal as in film or video. While the spatial dimension x and y can be continuous (film) or partly discrete (video where there is a line structure) the time dimension is always discrete (given by the number of images per second that are captured).

same wheel turning backward (or counter clockwise) by $-\pi/4$ leads to the same sampled sequence. This motion being smaller, it will be the "most likely" explanation that the viewer will see. The issue of aliasing in sampling will be studied in detail in Chapter 10.

While film and video is the most common and visible three-dimensional data set, there are many other examples of such signals, e.g. geophysical data (representation of the earth interior for oil exploration purposes), tomographic data (interior of the human body for medical analysis), etc. Finally, let us mention an example of a four-dimensional data set, used in medical signal processing. While its importance is clear, its acquisition is very difficult: it is the "image" of the beating heart. That is, using tomographic techniques, one reconstructs a three-dimensional image of the heart, and this over time.

In conclusion for this section on signals, let us simply remark that signals are everywhere. Wherever you look, listen or sense, signals are to be found. Humans are very good at processing signals for which they are equipped. But there are also many key signals beyond the "natural" ones, beyond the reach of humans. And all of these signals are of interest to signal processing systems.

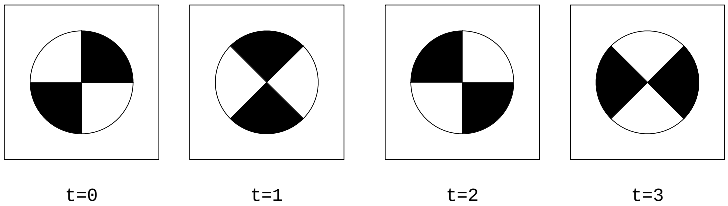


Figure 1.9: [illegible] or backward?

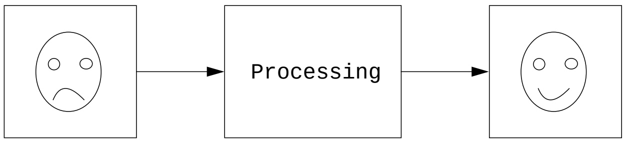


Figure 1.10: The "naive" view of a signal processing task: transforming a given signal (here a picture) into a desired signal.

1.4 Systems

A system, in our view of the world, is a box that takes a signal in, and produces an output, typically another signal. Such a general view, applied to the many signals we have seen, produces a wealth of possible signal processing systems. Instead of an exhaustive list, we will pick a few examples that are emblematic signal processing tasks. A "naive" picture of such systems is shown in Figure 1.10, where a given signal is transformed into a desired signal.

1.4.1 Speech Recognition

Given a speech signal, a speech recognition system tries to "understand" the words as a human would do. This seemingly elementary task (as seen from a human perspective) is actually dauntingly difficult for a computer in its full generality. While recognizing

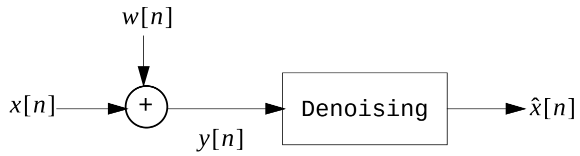


Figure 1.11: A signal $x[n]$ is corrupted by independent additive noise $w[n]$, followed by a denoising algorithm that produces an estimate $\hat{x}[n]$ given the observed signal $y[n] = x[n] + w[n]$.

individual elementary sounds spoken by a known speaker is easy, understanding a continuous stream of speech by an unknown speaker is still not solved satisfactorily as of today. Signal processing plays a key role in the first stages of a speech recognition system, that is in pre-processing (e.g. creating a compact representation of the speech waveform, for example through a local analysis of the spectrum) and in the modelization of speech (e.g. linear predictive models, hidden Markov models). After such signal processing based pre-processing, higher level methods are used (e.g. grammatical models for the structure of sentences). Speech recognition is even more complex in real environments, e.g. when noise is involved, as in a car for example.

1.4.2 Denoising

Very often, instead of getting the signal we want, we get a signal corrupted by noise. An obvious signal processing task is therefore to clean out the noise as well as possible without "damaging" the signal. To solve this problem, we need a model for the signal and the noise, so as to best estimate the signal given the observed signal. One such simple model is the additive noise model, where the noise is assumed to be independent of the signal, as shown in Figure 1.11. Many methods exist for attacking this problem, from filtering methods to non-linear denoising algorithms.

1.4.3 Inverse Problems

Numerous signal processing problems belong to the class of *inverse problems*. A generic example, which is also quite intuitive, is the following: assume you take a picture with your camera, but unfortunately, you take it out of focus. The result is a blurred image,

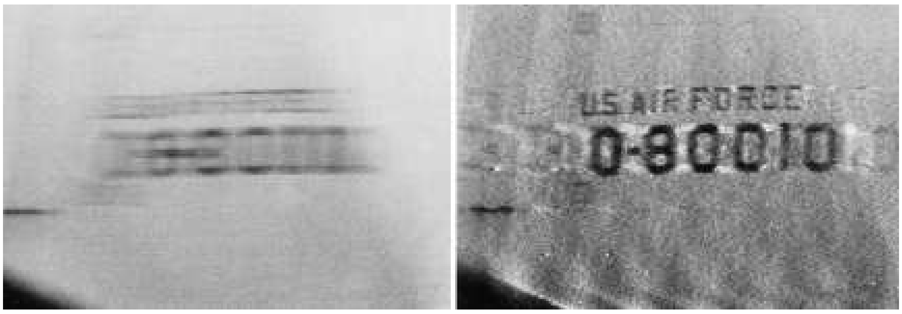


Figure 1.12: Example of an inverse problem. The original image (on the left) is blurred by the acquisition procedure; the image on the right is obtained digitally by "inverting" the blur operator (© Los Alamos National Laboratory)

and you would like to undo the blur and recover a sharp picture[1]. If and how to solve this is a typical inverse problem. In our example, the blurring operator is typically singular (which means certain components of the original image are forever lost) and thus its inverse is badly behaved (or ill-conditioned). Fig. 1.12 shows schematically the situation, where the inverse of the blur operator needs to be "regularized" so it is well-conditioned. There are many other instances of such inverse problems, like for example tomographic reconstruction in medical imaging or equalization for communication channels.

An additional reason why inverse problems are difficult is that very often, noise is present. That is, in our scheme shown in Figure 1.12, instead of y, we get $y + w$ where w is some noise signal. In that case, the inverse of the blur function can *amplify* the noise, and so, while the result might be sharper, it may also be very noisy. Then, combined "deblurring" and "denoising" is needed, a much more complex task.

1.4.4 Decision Systems

In many cases, a system takes a signal as its input, but produces just a binary output. For example, in the electrocardiogram case shown in Figure 1.7, a monitoring system simply needs to decide if the patient is healthy or not, but obviously, such a decision can be a matter of life or death. Similarly, a system monitoring the stock market index needs only to decide on buying or selling a particular stock. The characteristic of such systems is

[1]You may think that only total amateurs would run into such problems, but actually professionals can run into similar problems: NASA set up a rather expensive space telescope called Hubble, that was unfortunately "out of focus".

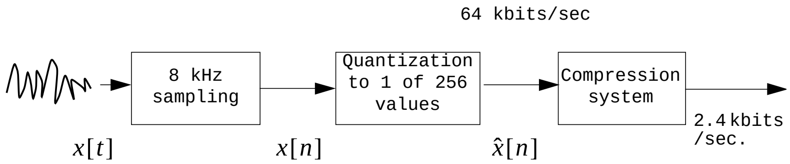


Figure 1.13: The original, continuous-time speech signal is first sampled at 8kHz, leading to a discrete-time signal $x[n]$. Each sample value is then approximated by 1 out of 256 values, and thus represented by an 8 bit number. This 64 kbits/sec digital stream is used in a complex compression system, that creates an approximate representation using only 2.4 kbits/sec.

that huge amounts of data are available, and all of it might be relevant to take the right decision. Such systems typically analyze time-series, and are thus under study in that particular field of statistics. Yet, similar problems exist in communication systems, where for example particular waveforms have to be detected, but are typically buried in noise.

1.4.5 Compression Systems

For storage and communication purposes, signals need to be represented by binary digits. That is, a discrete-time signal with real values $x[n] \in \mathbb{R}$ needs to be represented by a finite precision approximation so as to be representable by a binary number. For example, the voice samples are typically approximated using an 8 bit number, i.e 256 different values. This seems coarse, but is normally sufficient. But, beyond such simple sample by sample approximation, compression systems try to remove as much redundancy as possible from a given sampled and quantized representation. For speech, an original stream of 64 kbits/sec (corresponding to 8000 samples per sec.), each with an 8 bit representation) can be "compressed" to 8 kbits/sec, or even down to 2.4 kbits/sec, using sophisticated representation methods. A block diagram of such a compression system is shown in Figure 1.13.

Other well known compression systems are used for digital audio and video and such compression methods are key in all digital applications, from multi-media on CD-ROM's to video over the internet. The ubiquitous MP3 audio format, for example, is a sophisticated compression scheme which exploits a *perceptual model* of human hearing together with highly optimized quantization techniques. The perceptual model analyzes the audio input and determines which portions of the signal cannot be heard anyway due to masking effects (masking occurs when a strong spectral component "saturates" the ear around its frequency location, thereby making nearby components inaudible). Furthermore, the number of bits allotted to quantization is a time-varying quantity, determined so as to push the quantization noise below the masking threshold for the signal under analysis.

Image compression, on the other hand, exploits the high spatial redundancy of digital pictures and the fact that the eye is more sensitive to sharp edges than to color gradients. In the JPEG compression standard, the image is divided into a grid of square blocks and each block is processed individually. In the MPEG video compression standard (and in its derivative, DiVX), the former approach is complemented by a sophisticated prediction mechanism called *motion compensation*, so that the correlation between neighboring blocks in successive video frames is exploited to reduce the number of bits used to encode each image in the sequence of frames.

1.4.6 A Communication Systems Example

As an example of the ubiquity of signal processing in communication systems, consider Figure 1.14. Depicted is an interconnected system of networks of different kinds, with many different services that utilize signal processing in one way or another. As can be seen, signal processing is an enabling technology for communication systems, since it sits at the heart of the communication links (e.g. equalization, modulation), as well as at the heart of many applications, from voice to image and video communication, but also medical applications, multimedia databases, etc.

1.5 World Models

In many cases, we have prior knowledge about the signal we are processing or else, we can acquire knowledge about the signal as we process it. In both cases, there is a notion of a model behind the signal, and having good models for given signals is at the heart of signal processing. This leads to model based processing as shown in Figure 1.15. To be more specific consider a speech processing application. Speech (see Figure 1.1) is a very particular signal. It is usually produced by humans (we ignore for the moment talking parrots) and the speech production system is very well understood. Roughly speaking, speech is either voiced (in which case it has a harmonic or almost periodic structure) or unvoiced (that is noise-like). On top of this basic structure, the trachea, mouth, lip and nose filter the signal, producing a spectral shaping. Thus, an elementary speech production model is as shown in Figure 1.16. Now, any speech processing task can be helped by referring to

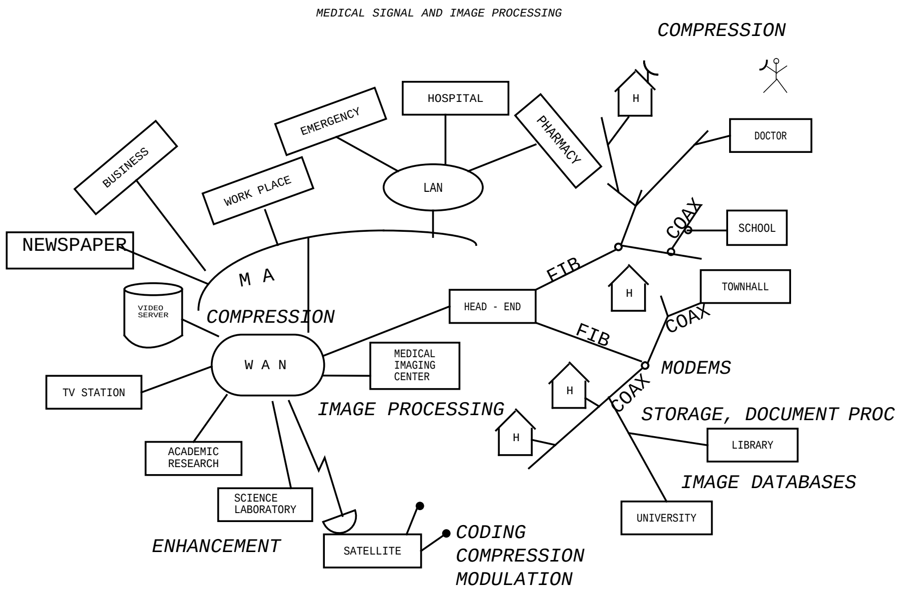


Figure 1.14: Signal processing in communication systems. In an interconnected world, with many applications using signals, signal processing is ubiquitous.

this model. In speech recognition, determining the voiced/unvoiced nature is a key task, as is the recovery of the fundamental frequency of the voiced part and the spectral shaping parameters. All these parameters feed into a ”pattern matching” algorithm that performs the actual recognition. In speech compression, recovering the fundamental parameters like voiced/unvoiced nature, pitch period, and spectral parameters leads to a very efficient representation, much more so than trying to approximate the original waveform. Finally, in speech synthesis (e.g. in text-to-speech synthesis systems), the model of Figure 1.16 is used to generate speech that sounds fairly natural.

Yet, there are potential problems with models. The first is complexity: models could become arbitrarily complex, thus difficult to estimate. The second is model mismatch: in our speech example, if the sound was actually from a parrot (unlikely but possible), one would have a hard time to find the parameters that are specific to humans!

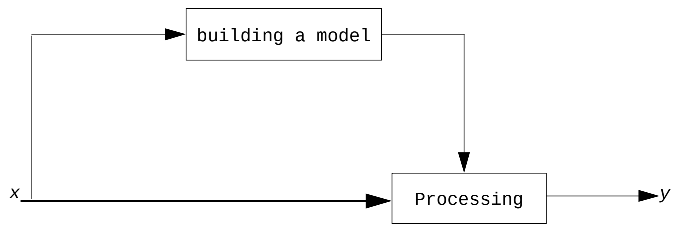


Figure 1.15: Model building and model based signal processing.

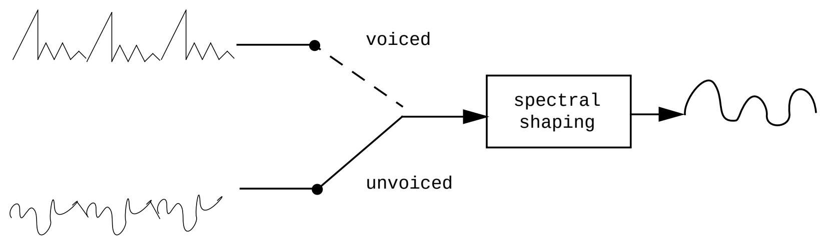


Figure 1.16: Basic speech production model, where two modes are considered (voiced/unvoiced) and time-varying spectral shaping is applied.

1.6 Analog and Digital Worlds

Signal processing is at the intersection of the analog and digital worlds. These worlds are very different, and are linked by sampling, quantization, and interpolation. After reviewing briefly these two worlds, we discuss advantages of each.

1.6.1 The Analog World (Continuous Time, Continuous Amplitude)

The world of analog signals is the world of functions on the real line, where the function is typically real valued. Thus, the time axis is continuous, and so is the amplitude. Most signals from the physical world are of this type: sound waves, electrical signals, physical measurements, etc. But also many man-made signals are of this type, like the output of a loudspeaker or the image on a video screen. Many systems from the physical world process such analog signals to produce other analog signals. For example, a physical communication channel takes an analog input (the signaling waveform) and produces an analog output (the received signal), even if the goal is to transmit a purely discrete information (like a bit from a file transfer). Analog signal processing, which has a long history, is typically performed with analog circuits that perform operations like filtering, amplifying, clipping etc. Analog filter design is a very well studied topic, with many "classic" designs. Nevertheless, it remains true that "good" analog filters are typically expensive, since they require high quality analog components. It's main advantage is that analog processing is "instantaneous" (at least in principle), that is, there is no time lag between input and output (other than phase factors or group delays).

1.6.2 Discrete-Time, Analog Worlds (Discrete Time, Continuous Amplitude)

In certain applications, continuous-time signals are sampled, but the real-valued samples are not further digitized (see next section). In that case, discrete-time analog circuits are used for the processing. An example is found in charge-coupled devices (CCD's), used for example in video cameras. While this type of processing corresponds mathematically to the "sampled system" case, it is rather the exception than the rule.

1.6.3 Digital Worlds (Discrete Time, Discrete Amplitude)

After sampling as in the previous section, the analog values are discretized to a countable set (typically a finite set). That is, both the time dimension and the amplitude dimension are now discrete. So we have the cascade of two operations:

$$x(t) \longrightarrow x[n] \longrightarrow \hat{x}[n]$$

the first being sampling, the second being quantization. The second operation cannot be undone, since quantization is a many-to-one mapping. Figure 1.17 shows the 3 types of signals we have just seen. Now, why would one give up the real, continuous amplitude world for this discretized and approximate representation? The reason lies of course in the fact that such discretized values can be represented in computer memory, and that if the discretization is fine enough, the representation is adequate. Because of the dominant position of digital computers in the technological world, the digital representation of signals is by now the most common. Note that both fixed point and floating point arithmetic[2]

[2] Fixed point arithmetic uses a fixed and finite set of values to represent amplitude, e.g. $[-N_1, -N_1 + 1, \ldots -1, 0, 1, \ldots N_2 - 1, N_2]$ where $N_i \in \mathbb{Z}$. The major problem is under and overflow during arithmetic operations. Floating point arithmetic uses a mantissa (similar to the fixed point numbers we just saw) and a scale factor given by an exponent with respect to the basis used (e.g. 2^k). While this allows a much

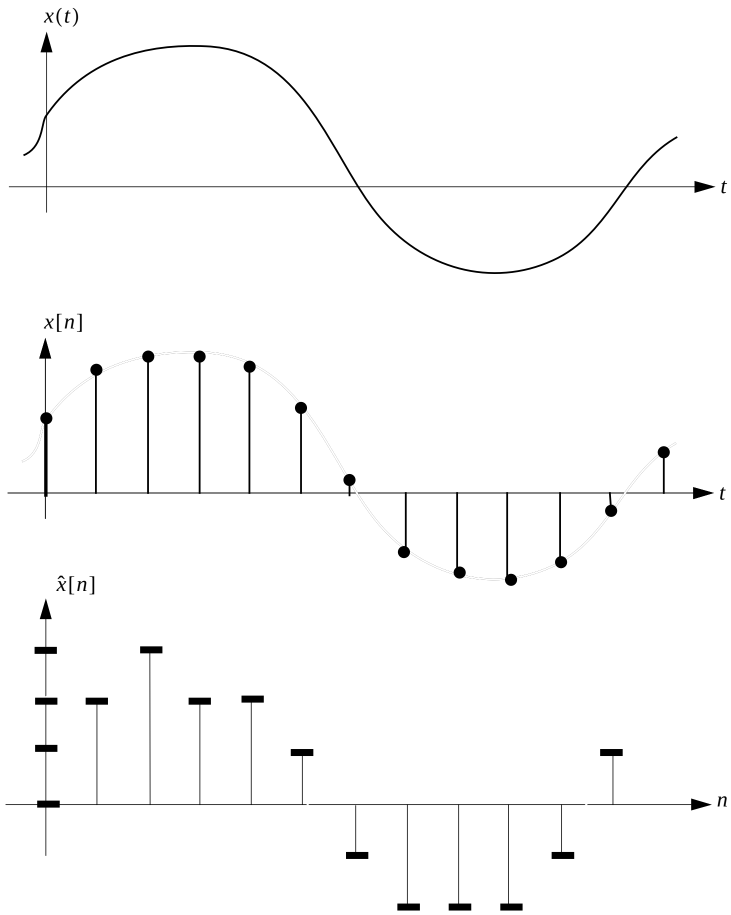


Figure 1.17: Various forms of signals between the analog and digital worlds.
(a) Analog, continuous-time signal. Both axes are continuous.
(b) Discrete-time, continuous-amplitude signal. The time axis is discrete.
(c) Discrete-time and quantized signal. Both axes are now discrete.

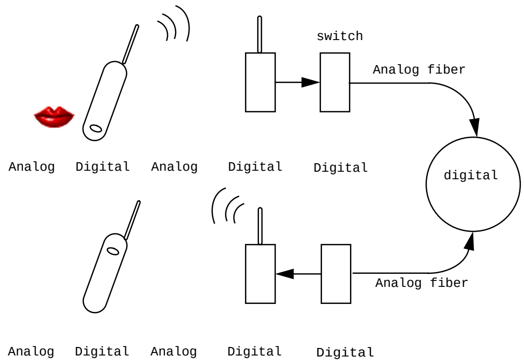


Figure 1.18: Multiple conversions between analog and digital representations in an end-to-end wireless phone communication.

are possible for such digital signal representation, but that both are discrete and finite amplitude representations.

1.6.4 Analog versus Digital

In many signal processing tasks, a key question is often: how much processing should be "analog", how much should be "digital". Take the design of a mobile phone system: clearly, input (voice) and output (loudspeaker) are analog, but inside, the system will probably go several times between analog and digital representations. Processing inside the mobile phone is digital, the digital communication over the wireless connection is analog, the base station converts back to digital, which goes over wireline backbone networks. From there, to reach another user, the reverse process is done, until an analog, acoustic signal is generated to reach the recipient's ear. Schematically, this is shown in Figure 1.18.

Another example is at the same time the best explanation of the amazing advance of digital communication systems and a paradigm for the pervasiveness of "digital" processing: analog versus digital telephony as seen over transatlantic links; this example is very simple and intuitive, but is at the heart of the digital "revolution". Given a transatlantic cable, should you use analog or digital transmission? In the analog case, you need repeaters, but these will boost signal and noise almost equally (See Figure 1.19-(a)). In

better approximation of the real numbers $\mathbb{R}$, it is still a finite representation, and underflows and round off errors are still problematic.

the digital case, there is an inital "noise" due to quantization, but then, as long as the noise added by the channel can be corrected by digital techniques (error correction codes as used in CD's), the noise gets annihilated, maintaining a reliable end-to-end quality (see Figure 1.19-(b)). The same phenomenon can be seen in copying analog signals (e.g. audio cassettes) versus digital signals (CD's). While a few cascaded copies from cassette to cassette will be too noisy to use, an arbitrary number of copies of CD's is no problem at all. (Except of course for the copyright owner !)

	Analog	Digital
+	World mostly analog Precision in principle ∞ Speed is arbitrary Arbitrary signals are possible	Digital computers dominate Calculations exact, reproducible Storage easy No noise
-	Computing mostly digital Noise difficult to control Storage difficult	World is analog, so need interface Initial imprecision due to quantization Speed limit Restricted signals

Table 1.2: Analog versus digital.

Finally, Table 1.2 summarizes some of the positive and negative points of analog versus digital representations and processing. From this table we can see that the comparison offers a mixed picture. Yet, in reality, the techniques of digital processing have advanced more and more and, at the same time, the devices used to implement digital signal processing algorithms have become more and more powerful and inexpensive. Today's standard desktop computers can easily perform in real time extremely complex tasks such as decoding DVD data, compress voice for internet telephony and modulate data for dial-up connections, and often in parallel. What's more, the time required for the industry to *develop and test* such algorithms is immensely inferior to what would be necessary to design their analog counterparts, admitting that that were at all possible. The global picture is that of an increasingly digital world with analog processing confined to the extreme boundaries, i.e. to the places where an interface to the physical mediums is necessary. This is why the stress of this course is on discrete-time processing techniques.

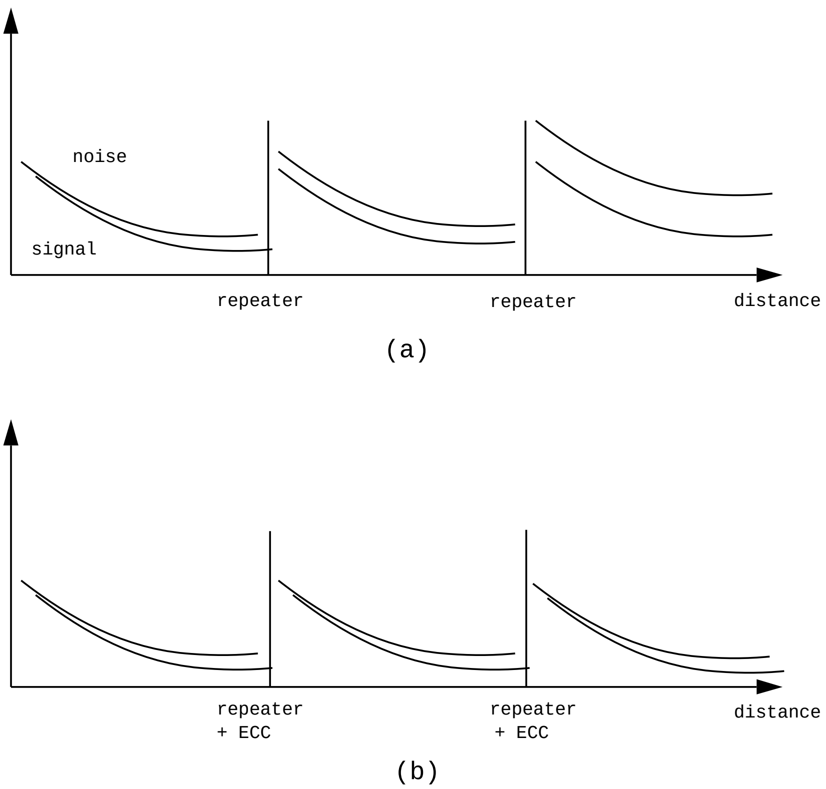


Figure 1.19: Comparison between analog and digital transmission over a transatlantic cable. (a) In the analog case, both the signal and the noise are amplified at the repeaters; (b) In the digital case, if the noise is not too much at the first repeater, a perfect reconstruction can be achieved, and the same holds for subsequent repeaters: the noise does not grow.

1.7 Overview of the Course

In this chapter we presented a general overview of signal processing and we tried to show the broad range of signals and systems where signal processing methods can be used. Interestingly, a number of methods are common to this vast array of applications, and these are at the center of our study. The rest of the course will try to lay a solid foundation for the mathematical basis of signal processing and proceed from there to illustrate more in details desigh techniques and applications. Here is a short description of each of the next chapters:

Chapter 2 will introduce more formally the classes of discrete-time signals which we will use in the course. It will also give an informal description of the sampling theorem that describes how to obtain discrete-time signals from continuous-time signal without information loss.

Chapter 3 will explain the different representations of periodic and finite block-lenght discrete-time sequences. Representations in terms of the discrete-time Fourier series (DFS) and the discrete Fourier transform (DFT) are introduced.

Chapter 4 will review of background material from applied mathematics and linear algebra, with an emphasis on geometric intuition via the concept of Hilbert spaces. In particular, this background is useful in extending the representations in Chapter 3 to infinite length discrete-time sequences.

Chapter will be devoted to the discrete-time Fourier transform (DTFT) which is a representation of discrete-time sequences. We will also study the properties of DTFT and relate it to the DFS and DFT studied in Chapter 3.

Chapter 6 will develop the ideas of how the Fourier representations can be used in practice. In particular, applications to spectral analysis, time-frequency analysis etc., are introduced.

Chapter 7 will introduce the notion of system, with an emphasis on linear time invariant systems. We will introduce the concept of convolution sum and its application to filtering, both in the time and the frequency domain. The concepts of stability and causality of a system are also introduced.

Chapter 8 will introduce the z-transform and its properties. The z-transform is the generalization of the Fourier transform to the complex plane in the discrete-time domain, just as the Laplace transform is the generalization of the Fourier transform in the continuous-time domain.

Chapter 9 will study the problem of digital filter design and filter implementation, with particular emphasis on FIR filters.

Chapter 10 deals with the fundamental operation of sampling, by which a continuous-time signal is converted into a discrete-time sequence, and interpolation, by which a discrete-time signal is converted to a continuous-time signal.

Chapter 11 will develop a generalization of standard, single sampling rate signal processing to deal with multiple sampling rates, which could be needed for sampling rate conversions. We develop the basic concepts of this rich topic by introducing filterbanks, sub-band decomposition and basic ideas of wavelets.

Chapter 12 will tackle quantization, or approximate representations. In particular, the problem of analog-to-digital conversion is studied in detail, including oversampling. We also study analog-to-digital conversion and its counterpart, digital-to-analog conversion.

Chapter 13 contains some application ideas such as denoising and a small project on multicarrier communications.

Appendix 1.A Historical Notes

The roots of signal processing are to be found in mathematics and applied mathematics, but the driving force behind its development lies in technological advances.

The idea of signal processing is probably as old as science itself (e.g. prediction of an eclipse based on past observations). Closer to us, the founding father of harmonic analysis, Joseph Fourier (1768-1830), is usually considered an important historical figure, given that Fourier series are central in signal processing. The importance of Fourier analysis is due to several facts, including the eigenfunction property of complex exponentials in linear time-invariant systems and the orthogonal expansion given by Fourier bases. Convergence questions (e.g. the Gibbs phenomenon) are also still important today, as they were when leading mathematicians questioned Fourier's claim that any periodic function could be written as a linear combination of harmonic sine and cosine waves.

A cornerstone result for discrete-time signal processing is the sampling theorem, often attributed to Shannon. While the theorem is indeed shown in Shannon's 1948 landmark paper on communication theory , it was due earlier to Whittaker and Kotelnikov. That bandlimited functions can be represented uniquely by its samples taken at twice the maximum frequency is one example of an interpolation formula, in this case using the *sinc* function. Another view of this result is that *sinc* functions and their translates form an orthonormal basis for the space of bandlimited functions. This view has been generalized recently with the theory of wavelets.

The sampling theorem of the mid 1940's lead to sampled data systems, which allowed to use the first computers for applications like ballistics. Such military applications also motivated Wiener to study the problem of prediction and noise removal, leading to Wiener filtering and its variants, some of which are still in use today. The key is that sampled systems allow the use of computers to implement sophisticated algorithms.

The 1950's and 1960's saw a lot of theoretical and practical research in fields like speech analysis and synthesis (e.g. the Vocoder) and very early image processing.

But the real "digital signal processing" revolution starts in 1965, with the publication by Cooley and Tukey of an efficient algorithm for the computation of Fourier series, the fast Fourier transform (FFT). Together with the availability of better computers and dedicated hardware, signal processing became a reality outside of the laboratory setting where it was confined until then. In particular, digital signal processing is a key element in making communication systems move into the digital age: speech moves from analog to digital, digital communications uses equalization, teleconferencing is demonstrated, etc.

The late 1960's and the following two decades are periods of intense developments. In particular, image and later video processing come of age, for example with the definition of compression standards that allow digital communication of images and moving pictures. But also medical imaging, with tomography and echography moving from research to actual usage, is revolutionized by digital processing. The most audible digitalization is of course the introduction of compact discs in 1984, which in a short time replaced analog records by a digital version.

In parallel, the microprocessor revolution leads to the introduction of specialized machines called signal processors, that is, microprocessors optimized for signal, image or even video processing tasks. Specialized chips using VLSI technology also become common place for particular, high end signal processing tasks.

The end of the 20th century sees signal processing everywhere, from personal computers with multimedia capabilities to complete, end-to-end digital communication for all forms of signals. While many of the techniques studied in the last decades are now part of everyday objects (e.g. portable phones), new questions constantly arise. To make an exhaustive list is difficult as well as a moving target, but a few topics that come to mind are the following:

- fundamental limits in compression
- joint source-channel coding, e.g. for channels like internet and mobile
- security issues, like watermarking of signals for copyright protection

- recognition, especially for very large data sets, e.g. image databases.

While standard signal processing tools are well understood, new tools are needed for some of the challenges listed above.

Appendix 1.B Literature

Because signal processing is both a popular and a useful topic, it has spawned many publications. An exhaustive list of books would take pages, and therefore we will only mention a subset that is either directly related to our notes, or which we feel is part of the general culture around signal processing.

On basic signal processing, there are several good books, including the recommended textbook for the class, which is *Discrete-Time Signal Processing*, by A. V. Oppenheim and R. W. Schafer (Prentice-Hall, 1989). As advanced signal processing, we can mention *A Wavelet Tour of Signal Processing*, by S. Mallat (Academic Press, 1999), and *Wavelets and Subband Coding*, by M. Vetterli and J. Kovacevic (Prentice Hall, 1995).

For background in signals and systems, we suggest *Signals and Systems* by Oppenheim, Wilsky and Nawab (Prentice Hall, 1997). For Fourier analysis, a nice engineering book is *The Fourier Transform and Its Applications*, by R. Bracewell (McGraw-Hill, 1999), while there are many mathematics textbooks on the topic such as *Fourier Analysis*, by T. W. Korner (Cambridge University Press, 1989).

The research literature on signal processing is published by the Institute of Electronics and Electrical Engineers (IEEE) in several transactions, but mainly in the one on Signal Processing, on Image Processing, and on Speech and Audio Processing. Several for profit publishers also run signal processing journals (e.g. Elsevier's *Signal Processing* magazine).