Chapter 10, Sampling and Interpolation: Problem Solutions

Problem 1

(a) Since $\tilde{F}(j\Omega)$ is periodic with period $2\pi/\Delta$, it admits a Fourier series representation as

$$\tilde{F}(j\Omega) = \sum_{k=-\infty}^{\infty} A_k e^{j\Delta k\Omega}. \tag{1}$$

Note that here, we are operating "backwards", i.e. the roles of time and frequency are reversed. The Fourier coefficients are computed as

$$\begin{aligned} A_k &= \frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} \tilde{F}(j\Omega)e^{-j\Delta k\Omega}d\Omega \\ &= \frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} \sum_{n=-\infty}^{+\infty} F(j(\Omega + \frac{2\pi}{\Delta}n))e^{-j\Delta k\Omega}d\Omega \end{aligned}$$

by inverting integral and summation

$$= \frac{\Delta}{2\pi}\sum_n \int_{-\pi/\Delta}^{\pi/\Delta} F(j(\Omega + \frac{2\pi}{\Delta}n))e^{-j\Delta k\Omega}d\Omega$$

and with a change of variable $\Omega' = \Omega + \frac{2\pi}{\Delta}n$

$$= \frac{\Delta}{2\pi}\sum_n \int_{(2n-1)\pi/\Delta}^{(2n+1)\pi/\Delta} F(j\Omega')e^{-j\Delta k\Omega'}e^{j\Delta\frac{2\pi}{\Delta}nk}d\Omega',$$

where $e^{j\Delta\frac{2\pi}{\Delta}nk} = 1$. These integrals are on contiguous, non-overlapping intervals, therefore

$$\begin{aligned} &= \Delta\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\Omega')e^{-j\Delta k\Omega'}d\Omega' \\ &= \Delta f(-k\Delta) \end{aligned}$$

so that by replacing the values for the A_k's in (1) we obtain $\tilde{F}(j\Omega) = \hat{F}(j\Omega)$.

(b) The above result shows that the proposed approximation is not a good idea in the general case, since the resulting periodization of the Fourier transform is not what we had in mind. This phenomenon is called *aliasing* and will be studied in detail in the context of the sampling of continuous-time signals.

(c) The three different cases are depicted in Figure 1.

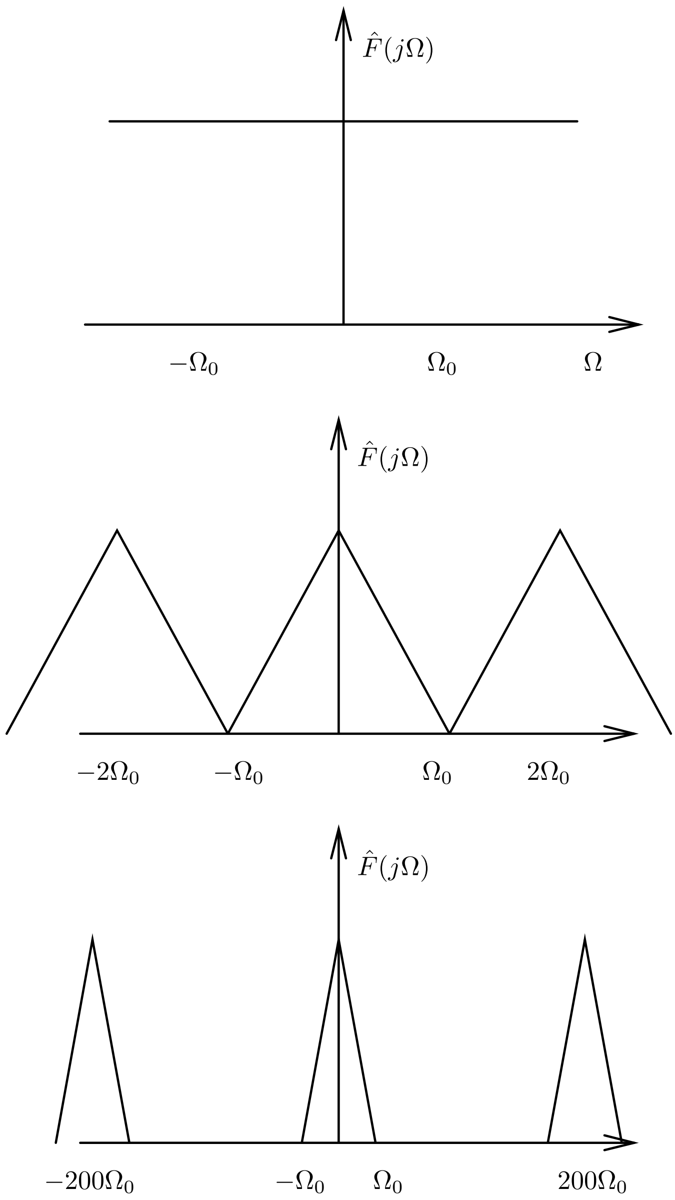


Figure 1: Aliasing for $\Delta = 2\pi/\Omega_0$, π/Ω_0 and $\pi/(100\Omega_0)$, respectively.

Problem 2

We have that

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

and, by using the modulation theorem,

$$\begin{aligned} X_s(j\Omega) &= X(j\Omega) * P(j\Omega) \\ &= \int_{\mathbb{R}} X(j\tilde{\Omega}) P(j(\Omega - \tilde{\Omega})) d\tilde{\Omega} = \frac{2\pi}{T_s} \int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k \in \mathbb{Z}} \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} \\ &= \frac{2\pi}{T_s} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} X(j\tilde{\Omega}) \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} = \frac{2\pi}{T_s} \sum_{k \in \mathbb{Z}} X\left(j\left(\Omega - k\frac{2\pi}{T_s}\right)\right). \end{aligned}$$

In other words, the spectrum of the delta-modulated signal is just the periodic repetition (with period $(2\pi/T_s)$ of the original spectrum. If the latter is bandlimited to (π/T_s) there will be no overlap and therefore $x(t)$ can be obtained simply by lowpass filtering $x_s(t)$ (in the continuous-time domain).

Problem 3

(a) According to our definition of bandlimited functions, the highest nonzero frequency is $2\Omega_0$ and therefore $x_c(t)$ is $2\Omega_0$-bandlimited for a total bandwidth of $4\Omega_0$. The maximum sampling period (i.e. the inverse of the *minimum* sampling frequency) which satisfies the sampling theorem is therefore $T_s = \pi/(2\Omega_0)$. Note however that the total support over which the (positive) spectrum is nonzero is the interval $[\Omega_0, 2\Omega_0]$ so that one could say that the total *effective* positive bandwidth of the signal is just Ω_0; this will be useful later.

(b) The digital spectrum will be the rescaled version of the periodized continuous-time spectrum

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_0)).$$

The general term $X_c(j\Omega - j2k\Omega_0)$ is nonzero only for

$$\Omega_0 \leq |\Omega - 2k\Omega_0| \leq 2\Omega_0 \quad \text{for } k \in \mathbb{Z}.$$

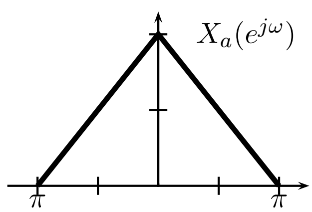
This translates to

$$\begin{aligned} (2k+1)\Omega_0 &\leq \Omega \leq (2k+2)\Omega_0 \\ (2k-2)\Omega_0 &\leq \Omega \leq (2k-1)\Omega_0 \end{aligned}$$

which are non-overlapping intervals! Therefore, there will be no disruptive superpositions of the copies of the spectrum. The digital spectrum will be simply

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s})$$

and it will look like this (with 2π-periodicity, of course):



(c) Here's a possible scheme (verify that it works):

- Sinc-interpolate $x_a[n]$ with period T_s to obtain $x_b(t)$
- Multiply $x_b(t)$ by $\cos(2\Omega_0 t)$ in the continuous time domain to obtain $x_p(t)$ (i.e. modulate by a carrier at frequency (Ω_0/π) Hz).
- Bandpass filter $x_p(t)$ with an ideal bandpass filter with (positive) passband equal to $[\Omega_0, 2\Omega_0]$ to obtain $x_c(t)$.

(d) The effective *positive* bandwidth of such a signal is $\Omega_\Delta = (\Omega_1 - \Omega_0)$. Clearly, the sampling frequency must be at least equal to the effective total bandwidth so we have a first condition on the maximum allowable sampling period: $T_{\max} < \pi/\Omega_\Delta$.

Now, to make things simpler, assume that the upper frequency Ω_1 is a multiple of the bandwidth, i.e. $\Omega_1 = M\Omega_\Delta$ for some integer M (in the previous case, it was $M = 2$). In this case, the argument we made in the previous point can be easily generalized: if we pick $T_s = \pi/\Omega_\Delta$ and sample we have that

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_\Delta)).$$

The general term $X_c(j\Omega - j2k\Omega_\Delta)$ is nonzero only for

$$\Omega_0 \leq |\Omega - 2k\Omega_\Delta| \leq \Omega_1 \quad \text{for } k \in \mathbb{Z}.$$

Since $\Omega_0 = \Omega_1 - \Omega_\Delta = (M-1)\Omega_\Delta$, this translates to

$$\begin{aligned} (2k + M - 1)\Omega_\Delta &\leq \Omega \leq (2k + M)\Omega_\Delta \\ (2k - M)\Omega_\Delta &\leq \Omega \leq (2k - M + 1)\Omega_\Delta \end{aligned}$$

which are again non-overlapping intervals.

If Ω_1 is *not* a multiple of the bandwidth, then the easiest thing to do is to change the lower frequency Ω_0 to a new frequency Ω_0' so that the new bandwidth $\Omega_1 - \Omega_0'$ divides Ω_1 exactly. In other words we set a new lower frequency Ω_0' so that it will be $\Omega_1 = M(\Omega_1 - \Omega_0')$ for some integer M; it is easy to see that

$$M = \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor.$$

since this is the maximum number of copies of the Ω_Δ-wide spectrum which fit *with no overlap* in the $[0, \Omega_0]$ interval. Note also that, if $\Omega_\Delta > \Omega_0$ we cannot hope to reduce the sampling frequency and we have to use normal sampling. This artificial change of frequency will leave a small empty "gap" in the new bandwidth $[\Omega_0', \Omega_1]$, but that's no problem. Now we can use the previous result and sample with $T_s = \pi/(\Omega_1 - \Omega_0')$ with no overlap. Since $(\Omega_1 - \Omega_0') = \Omega_1/M$, we have that, in conclusion, the maximum sampling period is

$$T_{\max} = \frac{\pi}{\Omega_1} \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor$$

i.e. we can obtain a sampling frequency reduction factor of $\lfloor \Omega_1/(\Omega_1 - \Omega_0) \rfloor$.