Chapter 11, Multirate Signal Processing: Problem Solutions

Problem 1

(a) **down-sampler and filtering:** We denote the systems and signals on the left and right hand sides by indices L and R, respectively.

For the z-transform of the down-sampled signals on the left, we can write

$$U_L(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{-j\frac{2\pi}{N}k} z^{\frac{1}{N}}),$$

and after filtering

$$Y_L(z) = H(z)U_L(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{-j\frac{2\pi}{N}k} z^{\frac{1}{N}})H(z).$$

For the system on the right we have

$$\begin{aligned} U_R(z) &= X(z)H(z^N) \\ \Longrightarrow Y_R(z) &= \frac{1}{N}\sum_{k=0}^{N-1} U(e^{-j\frac{2\pi}{N}k} z^{\frac{1}{N}}) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{-j\frac{2\pi}{N}k} z^{\frac{1}{N}})H(\underbrace{(e^{-j\frac{2\pi}{N}k})^N}_{1}\underbrace{(z^{\frac{1}{N}})^N}_{z}) \\ &= Y_L(z) \end{aligned}$$

(b) **up-sampling and interpolation:** As in the part (a), we use the indices L and R for signals on the left and right hand sides, respectively.

The z-transform of the system on the left is

$$\begin{aligned} U_L(z) &= X(z)H(z), \\ Y_L(z) &= U_L(z^L) = X(z^L)H(z^L), \end{aligned}$$

while the output of the system on the right corresponds to

$$\begin{aligned} U_R(z) &= X(z^L) \\ Y_R(z) &= U(z)H(z^L) = X(z^L)H(z^L) = Y_L(z). \end{aligned}$$

This shows that both of the systems have the same output for identical input, and they are equivalent.

Problem 2

(a) $v[n]$ is the up-sampled version of the input. So,

$$v[n] = \begin{cases} x[n/L] & n = kL \text{ for } k \in \mathbb{Z} \\ 0 & \text{else.} \end{cases}$$

Therefore, for $y = sL$, $s \in \mathbb{Z}$ we have

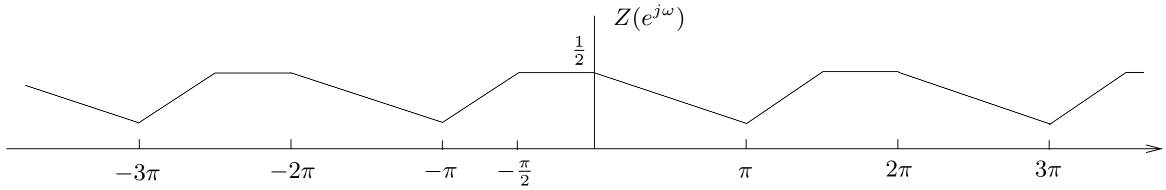
$$\begin{aligned} y[sL] &= h[sL] * v[sL] = \sum_{m=-\infty}^{\infty} h[m]v[sL-m] \\ &\overset{(*)}{=} h[0]v[sL] + 2\sum_{m=1}^{RL} h[m]v[sL-m] \\ &\overset{(**)}{=} h[0]x[s] + 2\sum_{k=1}^{R} h[(s-k)L]v[kL] \\ &= h[0]x[s] + 2\sum_{k=1}^{R} h[(s-k)L]x[k] \end{aligned}$$

which should be equal to $x[s]$, for any arbitrary input $x[n]$ and arbitrary integer s. In the equalities above, $(*)$ follows form the fact the $h[n] = h[-n]$ and $h[n] = 0$ for $|n| > RL-1$, and $(**)$ is true because $v[n]$ is non-zero only for $n = kL$, where $L \in \mathbb{Z}$. By feeding $x[n] = \delta[n]$, it is clear that $h[0] = 1$, and similarly, by feeding $x[n] = \delta[n+m-s]$ it follows that $h[mL] = 0$ for any $m \in \mathbb{Z}$.

(b) We can write

$$z[n] = y[2Ln] = x[2Ln/L] = x[2n],$$

which means $z[n]$ is the down-sampled version of $x[n]$ by down-sampling factor of 2.



Problem 3

(b) Recall the relationship between the spectrum of a continuous-time signal, the DTFT of the sampled version, and the FFT of the sampled version. It is known that if we sample from a continuous-time signal of bandwidth Ω_N at sampling frequency $f_s = \frac{1}{T_s}$, the original spectrum would be the repeated with period $2\pi f_s$ in the spectrum of the sampled version signal. The DTFT would be the same as the spectrum of the sampled signal, unless the ω-axis will be scaled by factor of T_s. This can be seen in Fig. 1. FFT is also can be considered as a discrete version of one period of DTFT as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}k} = \sum_{n=0}^{N-1} x[n]e^{-j\omega k}\Big|_{\omega=\frac{2\pi}{N}} = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}}$$

There fore one just has to find the cut tone form FFT, convert it to the cut frequency in DTFT and then using $\omega_n = 2\pi\frac{\Omega_N}{\Omega_s}$, in which $\Omega_s = 2\pi f_s$, where $f_s = 8192$ can be obtained for `[a,fs]=wavread('hw6.wav')`.

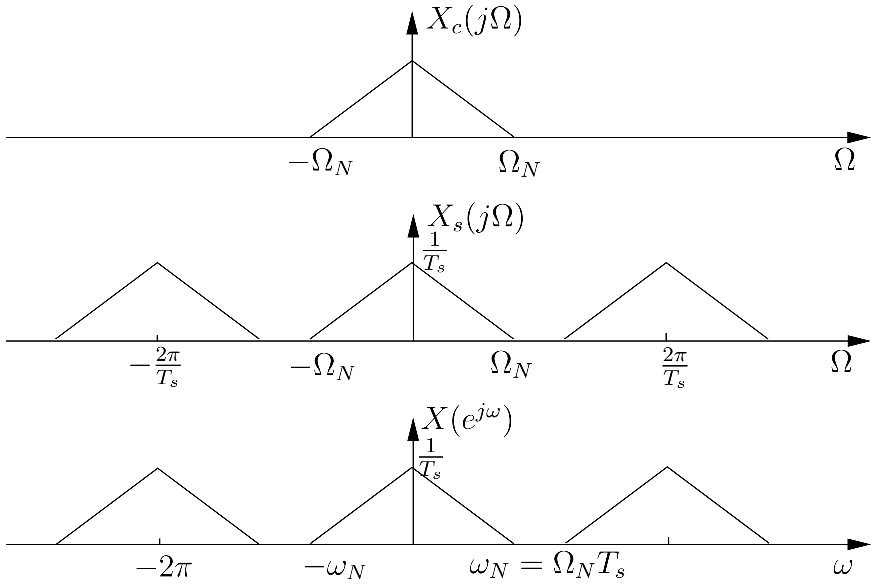


Figure 1: Relationship between the spectrum of a continuous-time signal and the sampled version.

The FFT of the signal is shown in Fig. 2 and which can be used to plot the approximated DTFT as in Fig. 3. For Fig.3 it is clear that the we have non-zero elements in the DTFT up to the frequency $\omega_N = \pi$ (bandwith in the minimum positive ω_N for which $X(e^{j\omega}) = 0$ for $\omega_N < |\omega| < \pi$). Therefore we have $\Omega_N = \omega_N f_s = 8192\pi$.

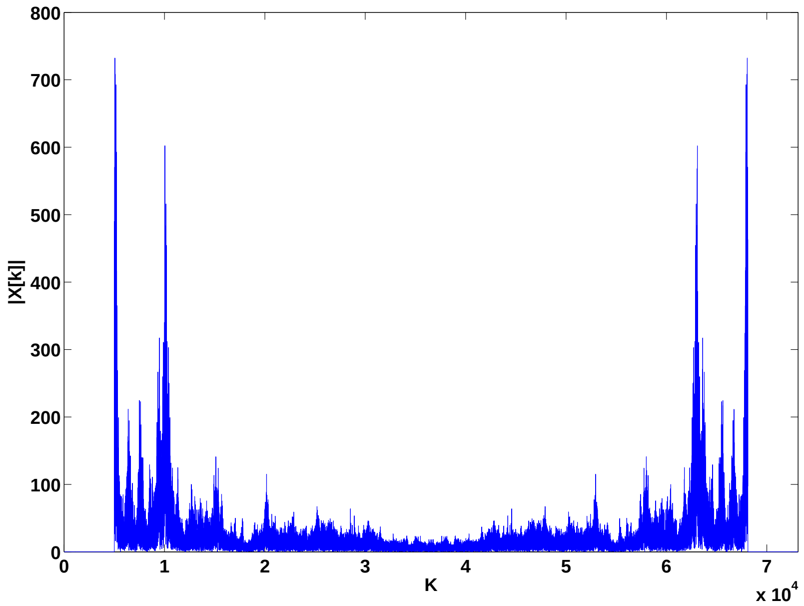


Figure 2: FFT of the signal

(c) The Matlab function `downsample(X,N,P)` downsamples input signal X by keeping every N-th sample starting with element in position $P+1$. So in order to produce $b[n] = a[3n]$, we can use

```
b=downsample(a,3,2);
```

(d) FFT of $b[n]$ is shown in Fig. 4.

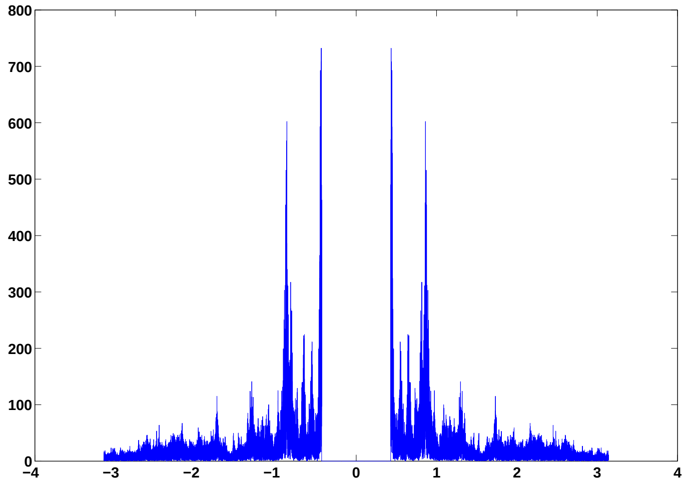


Figure 3: Approximated DTFT of the signal

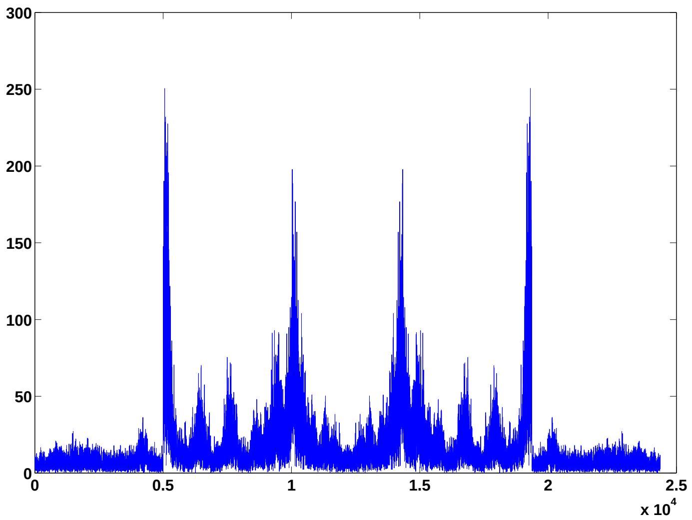
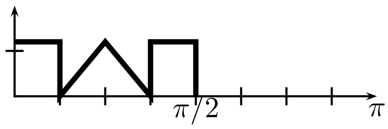


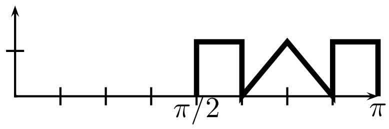
Figure 4: FFT of $b[n] = a[3n]$.

Problem 4

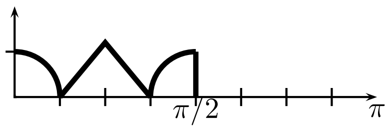
The spectrum of $y_1[n]$, $y_2[n]$, $y_3[n]$, and $y_4[n]$ are shown in the following figures.



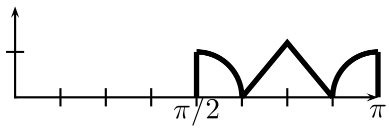
$Y_1(e^{j\omega})$



$Y_2(e^{j\omega})$



$Y_3(e^{j\omega})$



$Y_4(e^{j\omega})$

Problem 5

(a) We introduce $u[n]$ and $v[n]$ as illustrated in Figure 5. As done in class we introduce

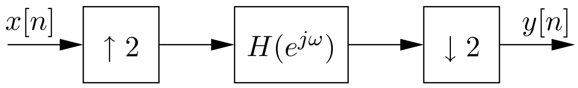
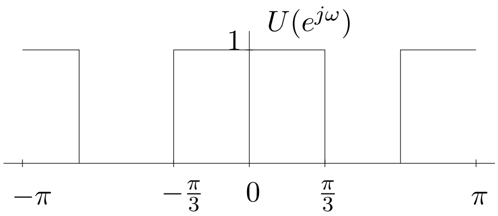


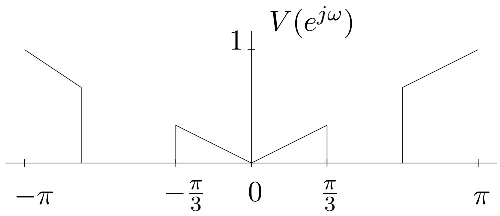
Figure 5: Problem 1. We introduce $u[n]$ and $v[n]$.

$$V'\left(e^{j\omega}\right) = \frac{1}{2}\left[V\left(e^{j\omega}\right) + V\left(e^{j(\omega-\pi)}\right)\right].$$

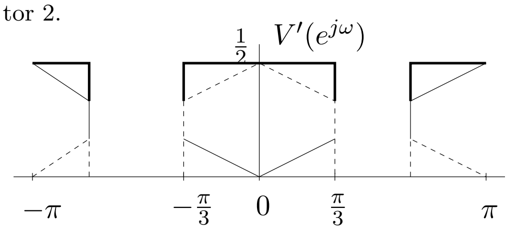
Based on this we have $Y(\left(e^{j\omega}\right) = V'\left(e^{j\omega/2}\right)$. We now develop $Y(\left(e^{j\omega}\right)$ in Figures 6(a) till 6(d).



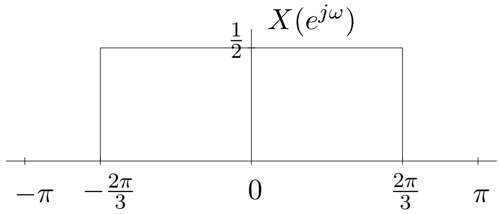
(a) $U(e^{j\omega})$, being $X(e^{j\omega})$ upsampled with a factor 2.



(b) $V(e^{j\omega})$, which is $U(e^{j\omega})$ filtered with $H(e^{j\omega})$.



(c) $V'(e^{j\omega})$. The thin solid line is $\frac{1}{2}V\left(e^{j\omega}\right)$, the dashed line is $\frac{1}{2}V\left(e^{j(\omega-\pi)}\right)$ and the thick line is $V'\left(e^{j\omega}\right)$.



(d) $Y(e^{j\omega})$, which we get by taking $Y(e^{j\omega}) = V'(e^{j\omega/2})$.

(b) We know from class that for upsampling we have the following relation

$$U\left(e^{j\omega}\right) = X\left(e^{j2\omega}\right).$$

Also after filtering we trivially have

$$V\left(e^{j\omega}\right) = X\left(e^{j2\omega}\right) H\left(e^{j\omega}\right).$$

After downsampling we get

$$\begin{aligned} Y((e^{j\omega}) &= \frac{1}{2}\left[V\left(e^{j\omega/2}\right) + V\left(e^{j(\omega/2-\pi)}\right)\right] \\ &= \frac{1}{2}\left[X\left(e^{j\omega}\right) H\left(e^{j\omega/2}\right) + X\left(e^{j(\omega-2\pi)}\right) H\left(e^{j(\omega/2-\pi)}\right)\right] \\ &= X\left(e^{j\omega}\right) \frac{1}{2}\left[H\left(e^{j\omega/2}\right) + H\left(e^{j(\omega/2-\pi)}\right)\right] \\ &= X\left(e^{j\omega}\right) G\left(e^{j\omega}\right), \end{aligned}$$

where we have used that $X((e^{j(\omega-2\pi)}) = X\left(e^{j\omega}\right)$, since the DTFT is 2π periodic. The thing to observe now is that $G(e^{j\omega}) = \frac{1}{2}\left[H((e^{j\omega/2}) + H((e^{j\omega/2-\pi})\right]$ is a downsampled version of the filter. In other words we have

$$g[n] = h[2n].$$

(c) We compute $G(e^{j\omega})$. The first step is to compute $\frac{1}{2}\left[H((e^{j\omega}) + H((e^{j\omega-\pi})\right]$, which we start in Figure 6. We immediately see that the after summing the two components we

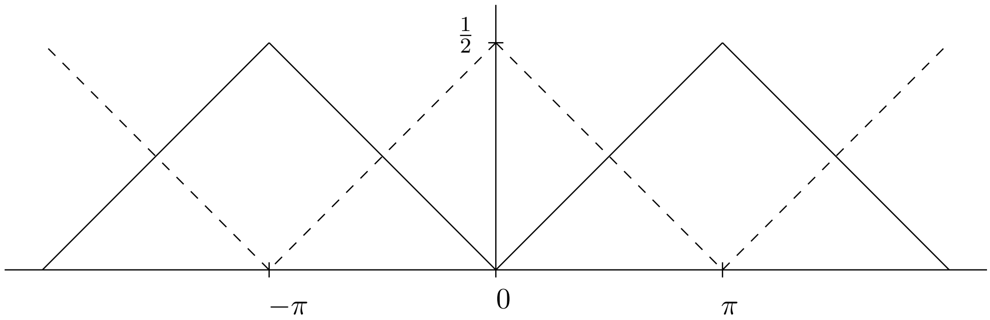


Figure 6: The first step in downsampling $H(e^{j\omega})$. In solid lines we have $\frac{1}{2}H\left(e^{j\omega/2}\right)$, in the dashed lines $\frac{1}{2}H\left(e^{j(\omega/2-\pi)}\right)$.

get

$$H'\left(e^{j\omega}\right) := \frac{1}{2}\left[H((e^{j\omega}) + H((e^{j\omega-\pi})\right] = \frac{1}{2}.$$

After scaling in frequency we have $G(e^{j\omega}) = H'\left(e^{j\omega/2}\right) = \frac{1}{2}$, which is an all-pass filter. If we now apply the result from Part (b) we have

$$\begin{aligned} Y\left(e^{j\omega}\right) &= X\left(e^{j\omega}\right) G\left(e^{j\omega}\right) \\ &= \frac{1}{2} X\left(e^{j\omega}\right), \end{aligned}$$

as was found in Part (a).

Problem 6

No solution available

Problem 7

No solution available