Chapter 5, The DTFT (Discrete-Time Fourier Transform): Problem Solutions

Problem 1

1. The inner product in $l_2(\mathbb{Z})$ is defined as

$$\langle x[n], y[n] \rangle = \Sigma_n x^*[n] y[n],$$

and in $L_2([-\pi, \pi])$ as

$$\langle X(e^{jw}), Y(e^{jw}) \rangle = \int_{-\pi}^{\pi} X^*(e^{jw}) Y(e^{jw}) dw.$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})Y(e^{jw})dw &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\Sigma_n x[n]e^{-jwn}\right)^* \Sigma_m y[m]e^{-jwm}dw \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\Sigma_n x^*[n]e^{jwn}\Sigma_m y[m]e^{-jwm}dw \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Sigma_n\Sigma_m x^*[n]y[m]e^{jw(n-m)}dw \\
&\overset{(2)}{=} \frac{1}{2\pi}\Sigma_n\Sigma_m x^*[n]y[m]\int_{-\pi}^{\pi}e^{jw(n-m)}dw \\
&\overset{(3)}{=} \Sigma_n x^*[n]y[n],
\end{aligned}$$

where (1) follows from the properties of the complex conjugate, (2) follows from swapping the integral and the sums and (3) from the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-m)}dw = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

2. If $x[n] = y[n]$, then $\langle x[n], x[n] \rangle$ corresponds to the energy of the signal in the time domain and $\langle X(e^{jw}), X(e^{jw}) \rangle$ to the energy of the signal in the frequency domain. In this case, the Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the *Parseval theorem*.

Problem 2

[DFT and DTFT]

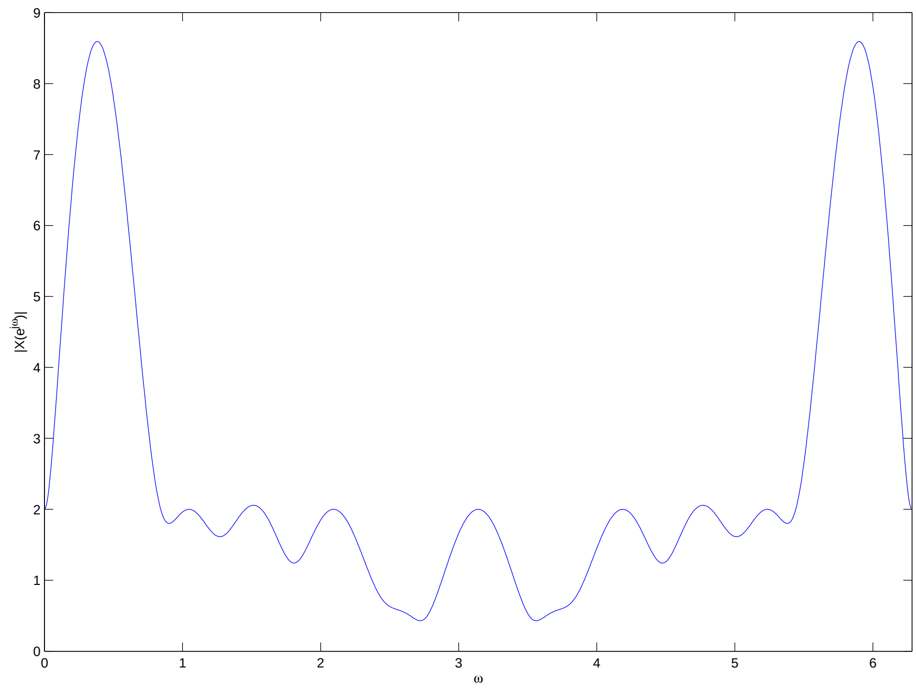

Figure 1: Problem 5(a).

(a)

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=3}^{9} e^{-j\omega n} - \sum_{n=10}^{14} e^{-j\omega n} \\
&= e^{-j\omega 3}\frac{1-e^{-j\omega 7}}{1-e^{-j\omega}} - e^{-j\omega 10}\frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} \\
&= \frac{e^{-j\omega 3} - 2e^{-j\omega 10} + e^{-j\omega 15}}{1-e^{-j\omega}},
\end{aligned}
$$

for $\omega \neq 0$. For $\omega = 0$ we have $X(e^{j0}) = \sum_{n=3}^{9} 1 - \sum_{n=10}^{14} 1 = 2$.

(b) See Figure 1 that was obtained by

```
>> w = linspace(0,2*pi,1e3+1);
>> w = w(1:end-1); % we don't want 2*pi itself
>> X = (exp(-i*w*3)-2*exp(-i*w*10)+exp(-i*w*15))./(1-exp(-i*w));
Warning: Divide by zero.
>> X(1) = 2;
>> plot(w,abs(X))
>> xlim([0 2*pi])
>> xlabel('\omega')
>> ylabel('|X(e^{j\omega})|')
```

(c)-(d) See Figure 2, where we give the result for $N = 100$ only:

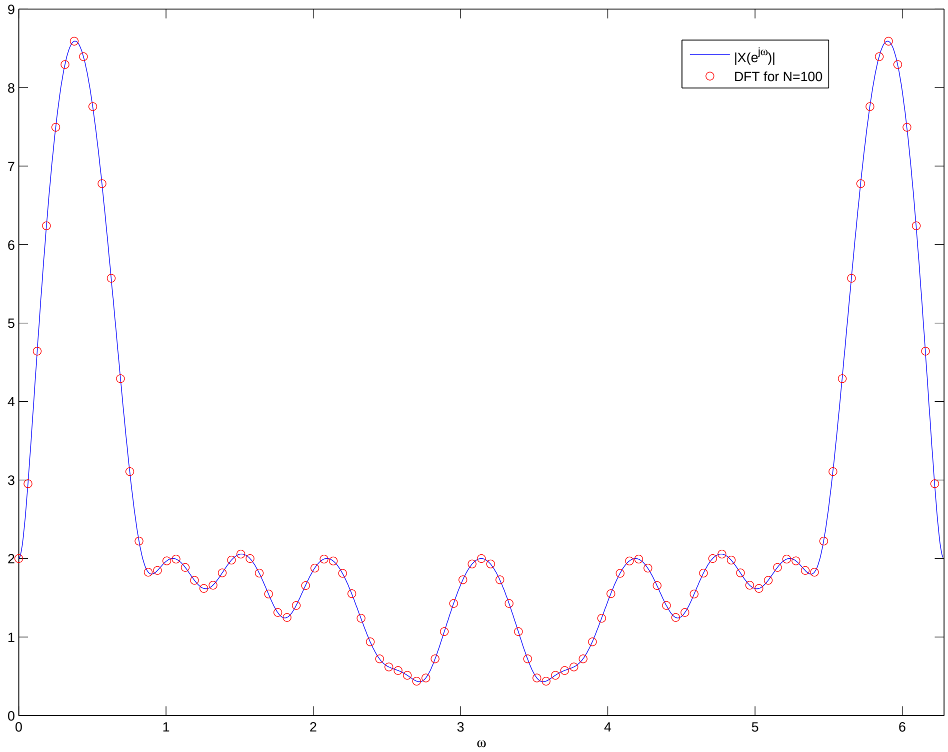

Figure 2: Problem 6(c)-(d).

```
>> N=1e2;
>> Xe2 = myDFT(x,N);
>> plot(w,abs(X))
>> xlim([0 2*pi])
>> hold on
>> plot([0:N-1]/N*2*pi,abs(Xe2),'or');
>> xlabel('\omega')
>> legend('|X(e^{j\omega})|','DFT for N=100');
```

We see that the the DFT sequence corresponds exactly to points on the DTFT curve.

Problem 3

1. The discrete-time sequence $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$x_1[n] = x_2[n] = \begin{cases} 1 & -(M-1)/2 \leq n \leq (M-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

In fact,

$$\begin{aligned} x_1[n] * x_2[n] &= \Sigma_k x_1[k] x_2[n-k] \\ &\stackrel{(1)}{=} \Sigma_k x_1[k] x_1[k-n] \\ &\stackrel{(2)}{=} x[n] \end{aligned}$$

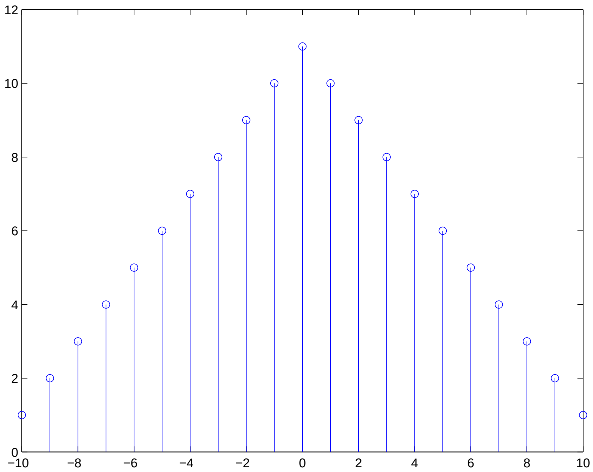

Figure 3: The discrete-time sequence $x[n]$ for $M = 11$.

where (1) follows from the fact that $x_1[n] = x_2[n]$ and from the symmetry of $x_1[n]$ and (2) noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its n-shifted version $x_1[k-n]$. When $|n| \geq M$ the two sequences do not overlap whereas the size of the overlapping area reaches its maximum M when $n = 0$.

Using Matlab, we can easily verify the above result for $M = 11$ using the following code:

```
>> M = 11;
>> x1 = ones(1,M);
>> x2 = x1;
>> x = conv(x1,x2);
>> stem([-M+1:M-1], x);
```

The result is shown in Figure 3.

2. Note that $x_1[n] = u[n+(M-1)/2] - u[n-(M+1)/2]$. We can thus compute its DTFT as

$$\begin{aligned} X_1(e^{j\omega}) &\overset{(1)}{=} \left(\frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)\right)\left(e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}\right) \\ &\overset{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})} \\ &= \frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned}$$

where (1) follows from the DTFT of $u[n]$ and (2) from the fact that

$$e^{jw(M-1)/2}\tilde{\delta}(w) = e^{-jw(M+1)/2}\tilde{\delta}(w) = \tilde{\delta}(w).$$

Using the convolution theorem, we can write

$$\begin{aligned} X(e^{jw}) &= X_1(e^{jw})X_2(e^{jw}) \\ &= X_1(e^{jw})X_1(e^{jw}) \\ &= \left(\frac{sin(\omega M/2)}{sin(\omega/2)}\right)^2. \end{aligned}$$

Problem 4

1. $\mathcal{H}\{\delta[n]\} = \delta[n]$; but $\mathcal{H}\{a\delta[n]\} = a^2\delta[n] \neq a\mathcal{H}\{\delta[n]\}$.

2. Let $y[n] = \mathcal{H}\{x[n]\}$; let $w[n] = x[n-n_0]$; $\mathcal{H}\{w[n]\} = w^2[n] = x^2[n-n_0] = y[n-n_0]$. QED.

3. First of all, $y[n] = \cos^2(\omega_0 n) = (1+\cos(2\omega_0 n))/2$ from the well-known trigonometric identity. So $y[n]$ contains a sinusoid at *double* the original frequency (but be careful: double in the 2π-periodic sense: if ω_0 is larger than $\pi/2$, then $2\omega_0$ will wrap around the $[-\pi,\pi]$ interval).

   If $\omega_0 = 3\pi/8$, then $y[n] = (1+\cos((3\pi/4)n))/2$; since $\mathcal{G}$ is a highpass with cutoff frequency $\pi/2$, it will kill the frequency components below $\pi/2$ and therefore it will kill the constant. The only component that passes through is the cosine at $3\pi/4$. The final output is therefore $v[n] = \frac{1}{2}\cos((3\pi/4)n)$.

4. If $\omega_0 = 7\pi/8$, then $2\omega_0 = 7\pi/4 > \pi$. We can therefore bring back the frequency into the $[-\pi,\pi]$ interval. We have that $7\pi/4 = 2\pi - \pi/4$ and therefore $\cos((7\pi/4)n) = \cos((2\pi-\pi/4)n) = \cos((\pi/4)n)$. So in the end $y[n] = (1+\cos((\pi/4)n))/2$. Now the frequency of the cosine is below $\pi/2$ and therefore $v[n] = 1+\cos((\pi/4)n)$. Note that, as for most nonlinear systems, the frequency of the input sinusoid is different from the frequency of the output sinusoids: sinusoids are no longer eigenfunctions!