# SOLUTIONS TO THE FINAL

## Instructions

- You are allowed to use 2 sheets of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- The $Q$-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2} du,$$
$$\text{and} \quad Q(-x) = 1 - Q(x).$$

- $Q(x) \leq e^{-\frac{x^2}{2}}$.
- For a $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- $\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$.
- You may find this property useful. If $h \sim \mathbf{C}\mathcal{N}(0,1)$,

$$\lim_{\delta \to 0} \left[ \frac{1}{\delta} \text{Prob}(|h|^2 < \delta) \right] = 1.$$

This means that for *small* $\delta$, $\text{Prob}(|h|^2 < \delta) \approx \delta$.

- You may find this lemma useful: If $U \sim \mathbf{C}\mathcal{N}(0, \mathbf{K}_U)$, and $\mathbf{A} = \mathbf{A}^*$, for some $b \in \mathbf{C}$ then

$$\mathbf{E}_U \left[ e^{-b\mathbf{U}^*\mathbf{A}\mathbf{U}} \right] = \frac{1}{|\mathbf{I} + b\mathbf{K}_\mathbf{U}\mathbf{A}|}.$$

- There are also 5 bonus points to be had making the maximum number of points 105/100.

Good Luck!

# Problem 1

[ NOISE CANCELLATION **(15pts)**]

**(a)** Orthogonality principle implies,

$$\mathbb{E}[(X - \hat{X})\mathbf{Y}^*] = 0$$

where $\mathbf{Y} = \begin{bmatrix} X + Z_1 \\ Z_2 \end{bmatrix}$. Let $\mathbf{W} = \begin{bmatrix} W_1, & W_2 \end{bmatrix}$, then orthogonality principle implies

$$\mathbb{E}[(X - \mathbf{W}\mathbf{Y})\mathbf{Y}^*] = 0$$

or,

$$\mathbf{W} = \mathbb{E}[X\mathbf{Y}^*](\mathbb{E}[\mathbf{Y}\mathbf{Y}^*])^{-1}$$

Since,

$$\mathbb{E}[\mathbf{Y}\mathbf{Y}^*] = \begin{bmatrix} \mathcal{E}_x + \sigma^2 & \rho\sigma^2 \\ \rho^*\sigma^2 & \sigma^2 \end{bmatrix}$$

and,

$$\mathbb{E}[X\mathbf{Y}^*] = \begin{bmatrix} \mathcal{E}_x, & 0 \end{bmatrix}.$$

We get that,

$$\mathbf{W}_{opt} = \begin{bmatrix} W_{1,opt}, & W_{2,opt} \end{bmatrix} = \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)}[1, \quad -\rho],$$

and

$$\widehat{X} = \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)}(X + Z_1 - \rho Z_2)$$

**(b)**

$$\sigma^2_{MMSE} = \text{Trace}\left(R_{XX} - W_{\text{opt}} R_{YX}\right).$$

We have,

$$R_{XX} = \mathcal{E}_x, \quad R_{YX} = \begin{bmatrix} \mathcal{E}_x \\ 0 \end{bmatrix}$$

So,

$$\sigma^2_{MMSE} = \frac{\mathcal{E}_x\sigma^2(1 - |\rho|^2)}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)}.$$

We get that for $|\rho| = 1, \sigma^2_{MMSE} = 0$. The interpretation is that if $|\rho| = 1$, $Z_1$ is perfectly predictable from $Z_2$ and hence can be canceled from $X + Z_1$. This will yield $\sigma^2_{MMSE} = 0$.

**(c)** The linear estimator for $Z_1$ from $Y_2$ is,

$$\hat{Z}_1 = WY_2.$$

Applying orthogonality principle we get that $\hat{Z}_1 = \rho Y_2 = \rho Z_2$. Thus $\hat{Y}_1 = Y_1 - \rho Y_2 = X + Z_1 - \rho Z_2$. Thus the best linear MMSE estimator can be again obtained by applying orthogonality principle.

$$W^{(2)}_{opt} = \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)},$$

and hence

$$\hat{X^{(2)}} = \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)}\hat{Y}_1 = \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2(1 - |\rho|^2)}(Y_1 - \rho Y_2),$$

which is the same as in (a). The interpretation is that it is optimal for the prediction of $X$ to optimally predict $Z_1$ from $Z_2$ and then cancel it from $Y_1$.

# Problem 2

[Inter-carrier Interference in OFDM **(25pts)**]

**(a)**

$$
\begin{aligned}
\mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{z} \\
\mathbf{F}\mathbf{y} &= \mathbf{F}\mathbf{S}\mathbf{P}\mathbf{x} + \mathbf{F}\mathbf{z} \\
\Rightarrow \quad \mathbf{Y} &= \mathbf{F}\mathbf{S}\mathbf{F}^*\mathbf{D}\mathbf{F}\mathbf{x} + \mathbf{F}\mathbf{z} \\
\Rightarrow \quad \mathbf{Y} &= \underbrace{\mathbf{F}\mathbf{S}\mathbf{F}^*\mathbf{D}}_{\mathbf{G}}\mathbf{X} + \mathbf{Z} \\
\Rightarrow \quad \mathbf{Y} &= \mathbf{G}\mathbf{X} + \mathbf{Z}
\end{aligned}
$$

**(b)**

$$
\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \dots, N-1,
$$

Hence,

$$
\text{SINR} = \frac{\mathbb{E}\left[|\mathbf{G}_{l,l}\mathbf{X}_l|^2\right]}{\mathbb{E}\left[\left|\sum_{q\neq l}\mathbf{G}_{l,q}\mathbf{X}_q\right|^2\right] + \mathbb{E}|Z_l|^2} = \frac{\mathcal{E}_x|\mathbf{G}_{l,l}|^2}{\mathcal{E}_x\sum_{q\neq l}|\mathbf{G}_{l,q}|^2 + \sigma_z^2}
$$

**(c)**

$$
\begin{aligned}
\mathbb{E}(\mathbf{Y}\mathbf{Y}^*) &= \mathbb{E}((\mathbf{G}\mathbf{X}+\mathbf{Z})(\mathbf{X}^*\mathbf{G}^*+\mathbf{Z}^*)) \\
&= \mathcal{E}_x\mathbf{G}\mathbf{G}^* + \mathbf{I}\sigma_z^2.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\mathbb{E}(\mathbf{X}_l\mathbf{Y}^*) &= \mathbb{E}(\mathbf{X}_l(\mathbf{X}^*\mathbf{G}^*+\mathbf{Z}^*)) \\
&= \mathbb{1}_l^T\mathcal{E}_x\mathbf{G}^*,
\end{aligned} \tag{2}
$$

where $\mathbb{1}_l^T = \begin{bmatrix} 0 & \cdots & \underbrace{1}_{l^{\text{th}}\text{position}} & 0 & \cdots & 0 \end{bmatrix}$. Orthogonality principle implies,

$$
\mathbb{E}((\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l)\mathbf{Y}^*) = 0
$$

$$
\Rightarrow \mathbb{E}(\mathbf{W}_l^*\mathbf{Y}\mathbf{Y}^*) = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)
$$

$$
\Rightarrow \mathbf{W}_l^* = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)(\mathbb{E}(\mathbf{Y}\mathbf{Y}^*))^{-1}
$$

Using equations 1,2 we get that,

$$
\Rightarrow \mathbf{W}_l^* = \mathcal{E}_x\mathbb{1}_l^T\mathbf{G}^*(\mathcal{E}_x\mathbf{G}\mathbf{G}^* + \mathbf{I}\sigma_z^2)^{-1}
$$

**(d)**

$$
\mathbf{G}_{l,q} = (\mathbf{F}\mathbf{S})_l(\mathbf{F}^*\mathbf{D})_q
$$

where $(\mathbf{FS})_l$ denotes the $l^{th}$ row of $\mathbf{FS}$ and $(\mathbf{F}^*\mathbf{D})_q$ denotes the $q^{th}$ column of $\mathbf{F}^*\mathbf{D}$.

$$\mathbf{G}_{l,q} = \frac{1}{N}\begin{bmatrix} e^{j2\pi f_0(N-1)} & e^{j2\pi f_0(N-2)}e^{-j\frac{2\pi}{N}(l-1)} \ldots & e^{j2\pi f_0(N-N)}e^{-j\frac{2\pi}{N}(l-1)(N-1)} \end{bmatrix}\begin{bmatrix} d_q \\ d_q e^{j\frac{2\pi}{N}(q-1)} \\ \vdots \\ d_q e^{j\frac{2\pi}{N}(q-1)(N-1)} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\sum_{p=1}^{N} e^{\left(j\frac{2\pi}{N}(q-l)-j2\pi f_0\right)(p-1)}$$

By using the summation formula for the geometric series we get,

$$\mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\left[\frac{1-e^{-j2\pi f_0 N}}{1-e^{j\frac{2\pi}{N}(q-l-f_0 N)}}\right] \quad \text{for} \quad f_0 \neq 0.$$

The ICI is significant when $\mathbf{G}_{l,q}$ is comparable to $\mathbf{G}_{l,l}$. When $f_0 N$ is large then this could occur, i.e., there is significant time variation over the block.

# Problem 3

[ MULTIPLE ACCESS CHANNEL AND TRANSMIT CODE **(30pts)**]

[*8pts*] **(a)** Let $\tilde{y_1}(k) = y_1(k)$ and $\tilde{y_1}(k+1) = -y_1^*(k+1)$, then

$$\tilde{y_1}(k) = y_1(k) = e_1 a_1 + e_2 a_2 + f_1 b_1 + f_2 b_2 + z_1(k)$$

$$\tilde{y_1}(k+1) = -y_1^*(k+1) = e_1^* a_2 - e_2^* a_1 + f_1^* b_2 - f_2^* b_1 + z_1(k+1)$$

From these two equations we get

$$\tilde{\mathbf{Y}}_\mathbf{1} = \begin{bmatrix}\tilde{y_1}(k) & \tilde{y_1}(k+1)\end{bmatrix} = \begin{bmatrix}a_1 & a_2\end{bmatrix}\begin{bmatrix}e_1 & -e_2^* \\ e_2 & e_1^*\end{bmatrix} + \begin{bmatrix}b_1 & b_2\end{bmatrix}\begin{bmatrix}f_1 & -f_2^* \\ f_2 & f_1^*\end{bmatrix} + \begin{bmatrix}\tilde{z}_1(k) & \tilde{z}_1(k+1)\end{bmatrix},$$

where $\tilde{z}_1(k) = z_1(k), \tilde{z}_1(k+1) = -z_1^*(k+1)$. Similarly by choosing $\tilde{y_2}(k) = y_2(k)$ and $\tilde{y_2}(k+1) = -y_2^*(k+1)$ we get,

$$\tilde{\mathbf{Y}}_\mathbf{2} = \begin{bmatrix}\tilde{y_2}(k) & \tilde{y_2}(k+1)\end{bmatrix} = \begin{bmatrix}a_1 & a_2\end{bmatrix}\begin{bmatrix}h_1 & -h_2^* \\ h_2 & h_1^*\end{bmatrix} + \begin{bmatrix}b_1 & b_2\end{bmatrix}\begin{bmatrix}g_1 & -g_2^* \\ g_2 & g_1^*\end{bmatrix} + \begin{bmatrix}\tilde{z}_2(k) & \tilde{z}_2(k+1)\end{bmatrix}.$$

[*8pts*] **(b)** Let

$$\begin{bmatrix}\check{\mathbf{Y}}_\mathbf{1} & \check{\mathbf{Y}}_\mathbf{2}\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{Y}}_\mathbf{1} & \tilde{\mathbf{Y}}_\mathbf{2}\end{bmatrix}\mathbf{W} = \begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}\begin{bmatrix}\mathbf{E} & \mathbf{H} \\ \mathbf{F} & \mathbf{G}\end{bmatrix}\mathbf{W} + \begin{bmatrix}\tilde{\mathbf{Z}}_\mathbf{1} & \tilde{\mathbf{Z}}_\mathbf{2}\end{bmatrix}\mathbf{W}.$$

Now,

$$\begin{bmatrix}\mathbf{E} & \mathbf{H} \\ \mathbf{F} & \mathbf{G}\end{bmatrix}\begin{bmatrix}\mathbf{I_2} & -\mathbf{E}^{-1}\mathbf{H} \\ -\mathbf{G}^{-1}\mathbf{F} & \mathbf{I_2}\end{bmatrix} = \begin{bmatrix}\mathbf{E}-\mathbf{H}\mathbf{G}^{-1}\mathbf{F} & -\mathbf{E}\mathbf{E}^{-1}\mathbf{H}+\mathbf{H} \\ \mathbf{F}-\mathbf{G}\mathbf{G}^{-1}\mathbf{F} & \mathbf{G}-\mathbf{F}\mathbf{E}^{-1}\mathbf{H}\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}}\end{bmatrix},$$

where $\tilde{\mathbf{H}} = \mathbf{E} - \mathbf{H}\mathbf{G}^{-1}\mathbf{F}, \tilde{\mathbf{G}} = \mathbf{G} - \mathbf{F}\mathbf{E}^{-1}\mathbf{H}$.

[*9pts*] **(c)** If

$$\mathbf{H} = \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix},$$

then,

$$\mathbf{H}^{-1} = \frac{1}{|h_1|^2 + |h_2|^2} \begin{bmatrix} h_1^* & h_2^* \\ -h_2 & h_1 \end{bmatrix},$$

In $\mathbf{H}^{-1}$, the relationship between the diagonal and cross-diagonal entries is the same as in $\mathbf{H}$. Hence $\mathbf{H}^{-1}$ has the same form as $\mathbf{H}$. The addition of $\mathbf{H}$ and $\mathbf{G}$ will again have the same form as $\mathbf{H}$ and $\mathbf{G}$. Now we need to show that the product $\mathbf{HG}$ has again the desired form.

$$\mathbf{HG} = \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix} \begin{bmatrix} g_1 & -g_2^* \\ g_2 & g_1^* \end{bmatrix} = \begin{bmatrix} h_1 g_1 - h_2^* g_2 & -(h_1 g_2^* + h_2^* g_1^*) \\ h_2 g_1 + h_1^* g_2 & -h_2 g_2^* + h_1^* g_1^* \end{bmatrix}$$

Now $(h_1 g_1 - h_2^* g_2)^* = -h_2 g_2^* + h_1^* g_1^*$ and $-(h_2 g_1 + h_1^* g_2)^* = -(h_1 g_2^* + h_2^* g_1^*)$. Hence the product $\mathbf{HG}$ also has the desired form. So the form of matrices is preserved under the operation of addition, multiplication and taking the inverse. Now $\tilde{\mathbf{H}} = \mathbf{E} - \mathbf{HG}^{-1}\mathbf{F}$ and $\tilde{\mathbf{G}} = \mathbf{G} - \mathbf{FE}^{-1}\mathbf{H}$, so both has the desired form.

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{h}_1 & -\tilde{h}_2^* \\ \tilde{h}_2 & \tilde{h}_1^* \end{bmatrix}, \tilde{\mathbf{G}} = \begin{bmatrix} \tilde{g}_1 & -\tilde{g}_2^* \\ \tilde{g}_2 & \tilde{g}_1^* \end{bmatrix}.$$

[*5pts*] **(d)** We have

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{h}_1 & -\tilde{h}_2^* \\ \tilde{h}_2 & \tilde{h}_1^* \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{H}}^* = \begin{bmatrix} \tilde{h}_1^* & \tilde{h}_2^* \\ \tilde{h}_2 & \tilde{h}_1 \end{bmatrix}$$

Then,

$$\tilde{\mathbf{H}}\tilde{\mathbf{H}}^* = \begin{bmatrix} \tilde{h}_1 & -\tilde{h}_2^* \\ \tilde{h}_2 & \tilde{h}_1^* \end{bmatrix} \begin{bmatrix} \tilde{h}_1^* & \tilde{h}_2^* \\ -\tilde{h}_2 & \tilde{h}_1 \end{bmatrix} = \begin{bmatrix} |\tilde{h}_1|^2 + |\tilde{h}_2|^2 & 0 \\ 0 & |\tilde{h}_1|^2 + |\tilde{h}_2|^2 \end{bmatrix} = (|\tilde{h}_1|^2 + |\tilde{h}_2|^2)\mathbf{I_2}$$

Hence,

$$\breve{\mathbf{Y}}_\mathbf{1}\tilde{\mathbf{H}}^* = ||\tilde{\mathbf{h}}||^2 \begin{bmatrix} a_1 & a_2 \end{bmatrix} + \breve{\mathbf{Z}}_\mathbf{1}\tilde{\mathbf{H}}^*.$$

Similarly we can show that

$$\breve{\mathbf{Y}}_\mathbf{2}\tilde{\mathbf{G}}^* = ||\tilde{\mathbf{g}}||^2 \begin{bmatrix} b_1 & b_2 \end{bmatrix} + \breve{\mathbf{Z}}_\mathbf{2}\tilde{\mathbf{G}}^*,$$

This completes the decoupling of the individual streams of the multiple access channel.

[*Bonus 5pts*] **(e)** Now the main point to note is the distribution of $\tilde{\mathbf{H}}$. Conditioned on $\mathbf{H}, \mathbf{G}$

$$\tilde{\mathbf{H}} = \mathbf{E} - \mathbf{HG}^{-1}\mathbf{F}$$

is Gaussian, with elements that are conditionally independent. If $||\mathbf{G}||^2 = |g_1|^2 + |g_2|^2$, we have

$$\begin{aligned} \mathbf{HG}^{-1} &= \frac{1}{||\mathbf{G}||^2} \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix} \begin{bmatrix} g^*{}_1 & g_2^* \\ -g_2 & g_1 \end{bmatrix} \\ &= \frac{1}{||\mathbf{G}||^2} \begin{bmatrix} h_1 g_1^* + h_2^* g_2, & h_1 g_2^* - h_2^* g_1 \\ h_2 g_1^* - h_1^* g_2, & h_2 g_2^* + h_1^* g_1 \end{bmatrix} \end{aligned}$$

and

$$\mathbf{H}\mathbf{G}^{-1}\mathbf{F} = \frac{1}{||\mathbf{G}||^2}\begin{bmatrix} h_1 g_1^* + h_2^* g_2, & h_1 g_2^* - h_2^* g_1 \\ h_2 g_1^* - h_1^* g_2, & h_2 g_2^* + h_1^* g_1 \end{bmatrix}\begin{bmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{bmatrix}$$

$$= \frac{1}{||\mathbf{G}||^2}\begin{bmatrix} h_1 g_1^* f_1 + h_2^* g_2 f_1 + h_1 g_2^* f_2 - h_2^* g_1 f_2, & -h_1 g_1^* f_2^* - h_2^* g_2 f_2^* + h_1 g_2^* f_1^* - h_2^* g_1 f_1^* \\ h_1^* g_1 f_2 + h_2 g_2^* f_2 - h_1^* g_2 f_1 + h_2 g_1^* f_1, & h_1^* g_1 f_1^* + h_2 g_2^* f_1^* + h_1^* g_2 f_2^* - h_2 g_1^* f_2^* \end{bmatrix}$$

Therefore

$$\begin{aligned} \mathbb{E}\left(|\tilde{h}_1^2|\right) &= 1 + \frac{|h_1 g_1^* + h_2 g_2^*|^2 + |h_1 g_2^* - h_2^* g_1|^2}{||\mathbf{G}||^2}, \\ &= 1 + \frac{||\mathbf{H}||}{||\mathbf{G}||}. \end{aligned}$$

Also $\mathbb{E}\left(|\tilde{h}_2|^2\right) = \mathbb{E}\left(|\tilde{h}_1|^2\right)$ and therefore conditioned on $\mathbf{H}, \mathbf{G}$ the matrix $\tilde{\mathbf{H}}$ is Gaussian. By scaling it with $\left(1 + \frac{||\mathbf{H}||}{||\mathbf{G}||}\right)^{-\frac{1}{2}}$ the matrix is Gaussian unconditionally as well. After this scaling we get the diversity order of 2 as done in the class.

# Problem 4

[Relay Diversity **(30pts)**]

[*6pts*] **(a)** Probability of error is given by

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1 + SNR}}\right)$$

Hence the diversity order, *i.e.*,

$$\lim_{\text{SNR}\to\infty} \frac{-\log(\bar{P}_e(\text{SNR}))}{\log(\text{SNR})} = 1.$$

[*12pts*] **(b)** By symmetry the error analysis is same for both the transmitted symbols, $a$ and $-a$. Hence we do the analysis when $a$ is transmitted.

The relay decodes correctly then the destination $\mathcal{R}$ uses both $y_{\mathcal{T}}^{(1)}(k)$ and $y_{\mathcal{T}}^{(2)}(k)$ for detection with out doing any thing. In this case the decoding is based on:

$$\begin{aligned} y_{\mathcal{T}}^{(1)}(k) &= ha + z_{\mathcal{T}}(k), \\ y_{\mathcal{T}}^{(2)}(k) &= ga + z_{\mathcal{T}}(k). \end{aligned}$$

On the other hand if the relay decodes incorrectly then $\mathcal{T}$ can negate the observation from $\mathcal{R}$. Now the decoding is based on:

$$\begin{aligned} y_{\mathcal{T}}^{(1)}(k) &= ha + z_{\mathcal{T}}(k), \\ -y_{\mathcal{T}}^{(2)}(k) &= -ga - z_{\mathcal{T}}(k). \end{aligned}$$

Now both $h$ and $g$ are independent, so the error analysis is same as in the case of repetition coding. This implies that the diversity order is 2.

[12pts] **(c)** The average probability of error is given by

$$\begin{aligned} P_e &= P(\mathcal{S} \text{ transmits in phase 2}) P(\text{error}|\mathcal{S} \text{ transmits in phase 2}) + \\ &\quad P(\mathcal{R} \text{ transmits in phase 2}) P(\text{error}|\mathcal{R} \text{ transmits in phase 2}). \end{aligned}$$

Now

$$P(\mathcal{S} \text{ transmits in phase 2}) = P(|f|^2 \leq \frac{1}{SNR^{1-\epsilon}}) \approx \frac{1}{SNR^{1-\epsilon}}.$$

When $\mathcal{S}$ transmits in phase 2, then the diversity order is

$$P(\text{error}|\mathcal{S} \text{ transmits in phase 2}) \approx \frac{1}{SNR}$$

The case when $\mathcal{R}$ transmits, its probability is

$$P(\mathcal{R} \text{ transmits in phase 2}) = 1 - \frac{1}{SNR^{1-\epsilon}}$$

The probability of error when $\mathcal{R}$ transmits depends on whether $\mathcal{R}$ commits an error. Let $\mathcal{R}_e$ denotes an error event at $\mathcal{R}$ and $\mathcal{R}_c$ denotes the event that $\mathcal{R}$ takes a correct decision.

$$\begin{aligned} P(\text{error}|\mathcal{R} \text{ transmits in phase 2}) &= P(\mathcal{R}_e) P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_e) \\ &\quad + P(\mathcal{R}_c) P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_c). \end{aligned}$$

But $\mathcal{R}$ only transmits when the channel condition is good. So $P(\mathcal{R}_e)$ should be small. In fact,

$$P(\mathcal{R}_e) \approx Q\left(\sqrt{SNR^{\epsilon}}\right) \leq e^{-\frac{SNR^{\epsilon}}{2}}.$$

Hence we can bound $P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_e)$ by 1 as the factor $Q\left(\sqrt{SNR^{\epsilon}}\right) \leq e^{-\frac{SNR^{\epsilon}}{2}}$ which is much smaller than $\frac{1}{SNR^{2-\epsilon}}$. Therefore

$$P(\mathcal{R}_e) P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_e) \leq Q\left(\sqrt{SNR^{\epsilon}}\right).$$

Also

$$\begin{aligned} P(\mathcal{R}_c) &= 1 - P(\mathcal{R}_e) \\ &= 1 - Q\left(\sqrt{SNR^{\epsilon}}\right) \\ \Rightarrow \quad P(\mathcal{R}_c) P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_c) &= \left(1 - Q\left(\sqrt{SNR^{\epsilon}}\right)\right) \frac{1}{SNR^2}. \end{aligned}$$

Hence,

$$\begin{aligned} P_e &\approx \frac{1}{SNR^{1-\epsilon}} \frac{1}{SNR} + \left(1 - Q\left(\sqrt{SNR^{\epsilon}}\right)\right) \frac{1}{SNR^2} + \\ &\quad Q\left(\sqrt{SNR^{\epsilon}}\right) P(\text{error}|\mathcal{R} \text{ transmits in phase } 2, \mathcal{R}_e). \end{aligned}$$

The last term is exponentially small in $SNR$. This implies that the diversity order is $2 - \epsilon$.