# Homework 5

## Problem 1

In a wireless setting with high scattering of the transmitted electromagnetic wave, the output of the channel can be modeled as follows.

$$y(t) = h(t)x(t),$$

where

$$h(t) = \sum_{l=1}^{L} h_l(t) = \sum_{l=1}^{L} C_l e^{j(2\pi \frac{v}{\lambda} \cos\theta_l t + \phi_l)},$$

where $C_l$ is a random amplitude of the $l$th incoming wave, $\theta_l$ is its random angle with respect to the direction of the moving receive-antenna, and $\phi_l$ is a random phase-shift. We will investigate the statistical properties of $h(t)$. For this, we assume that $C_l$,$\theta_l$ and $\phi_l$ are mutually independent, and each i.i.d. for $l = 1, \ldots, L$. We further assume that $C_l$ has second moment $\frac{\sigma^2}{L}$, that $\phi_l$ is uniformly distributed in $[0, 2\pi]$ and that $\theta_l$ is uniformly distributed in $[0, 2\pi]$. In a communication system, we will sample $y(t)$ at discrete time-steps. We will therefore analyze $h(t)$ at a fixed time instant $t_0$.

(a) For a given $t_0$, show that the expectation $\mathbb{E}[h_l(t_0)]$ is 0.
Hint: Use independence to write $h_l(t_0)$ as a product of expectations. It suffices to show that one of the factors is 0.

(b) Show that the variance of $h_l(t_0)$ is $\frac{\sigma^2}{L}$.

We will use the central limit theorem: If $X_i$, $i = 1, \ldots, L$ are i.i.d. random variables of any distribution with mean 0 and variance 1, then

$$\lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_{i=1}^{L} X_i \sim \mathcal{N}(0, 1),$$

i.e., the sum of $L$ i.i.d. random variables converges to a random object that has Gaussian distribution.

(c) Define $X_l = \frac{\sqrt{N}}{\sigma} h_l(t_0)$ and apply the central limit theorem to $X_l$. Conclude that $h(t_0)$ behaves, for large $L$, as a Gaussian random variable of zero mean and variance $\sigma^2$.

After passing $y(t)$ through a matched filter, only the amplitude (the absolute value) of $h(t)$ will matter. The absolute value of a Gaussian random variable has a Rayleigh distribution. The factor $|h(t_0)|$ is called the Rayleigh fading factor.