Solutions to Homework 4

Problem 1

(a) Firstly,

$$\begin{aligned} R'(D) &= R(D) + (1 - B(D))\,(X(D) - R(D))\,, \\ &= B(D)R(D) + (1 - B(D))\,X(D). \end{aligned}$$

Hence by using the orthogonality principle,

$$\mathbb{E}\left(E(D)Y^*\left(D^{-*}\right)\right) = 0$$

where

$$E(D) = X(D) - R'(D) = B(D)X(D) - B(D)R(D) = B(D)X(D) - B(D)H(D)Y(D).$$

Hence

$$\begin{aligned} B(D)S_{xy}(D) - B(D)H(D)S_{yy}(D) &= 0, \\ \Rightarrow H(D) &= S_{xy}(D)S_{yy}^{-1}(D), \\ &= \frac{||p||Q(D)\mathcal{E}_x}{||p||^2Q^2(D)\mathcal{E}_x + N_0Q(D)}, \\ &= \frac{1}{||p||\left(Q(D) + \frac{1}{\mathrm{SNR}_{\mathrm{MFB}}}\right)}. \end{aligned}$$

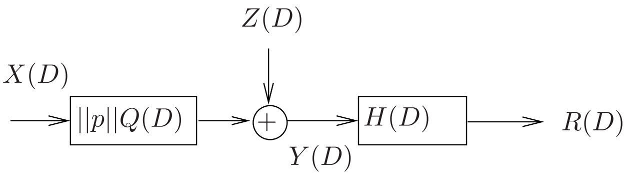
where $\mathrm{SNR}_{\mathrm{MFB}} = \frac{\mathcal{E}_x||p||}{N_0}$.

(b)

$$\begin{aligned} E(D) &= B(D)\underbrace{\left(X(D) - S_{xy}(D)S_{yy}^{-1}(D)Y(D)\right)}_{U(D)}, \\ \Rightarrow S_U(D) &= \mathbb{E}\big(U(D)U^*(D^{-*})\big) \\ &= \mathbb{E}\big(U(D)X^*(D^{-*})\big) + \mathbb{E}\big(U(D)S_{xy}^*(D^{-*})S_{yy}^{-*}(D^{-*})Y^*(D^{-*})\big) \\ &= \mathbb{E}\big(U(D)X^*(D^{-*})\big) + S_{xy}^*(D^{-*})S_{yy}^{-*}(D^{-*})\frac{1}{B(D)}\underbrace{\mathbb{E}\big(E(D)Y^*(D^{-*})\big)}_{=0} \\ &= S_{xx}(D) - S_{xy}(D)S_{yy}^{-1}(D)S_{xy}(D), \\ &= \mathcal{E}_x - \frac{||p||^2Q^2(D)\mathcal{E}_x^2}{||p||^2Q^2(D)\mathcal{E}_x + N_0Q(D)}, \\ &= \frac{N_0/||p||^2}{Q(D) + \frac{1}{\mathrm{SNR}_{\mathrm{MFB}}}}. \end{aligned}$$

One can observe that $H(D) = W_{MMSE-LE}(D)$ as derived in class. If $B(D) = 1$, then the D-transform of the error $E(D) = B(D)$ is the same as in $MMSE - LE$.

(c) If we add the channel part, the figure is as follows:



Problem 2

(a) $Q(D)$ is given by:

$$Q(D) = 1 + bD + bD^{-1}$$

The factorization must be of the following form:

$$Q(D) = v_o P_c(D) P_c^*(D^{-*}) = \beta(1 + \alpha D)(1 + \alpha D^{-1})$$

Developing and Identifying we find: $\beta = \frac{1 \pm \sqrt{1-4b^2}}{2}$ and $\alpha = \frac{b}{\beta}$. With $b = \frac{1}{2}$ we get the following factorization for $Q(D)$:

$$Q(D) = \frac{1}{2}(1 + D)(1 + D^{-1})$$

(b)The derivation of ZF-DFE is equivalent to MMSE-LE DFE with $SNR_{\text{MFB}} = \infty$. Therefore, $v_0 = \frac{1}{2}$ and $P_c(D) = 1 + D$. This implies

$$B(D) = P_c(D) = 1 + D,$$

$$W(D) = \frac{1}{v_0 ||p|| P_c^*(D^{-*})} = \frac{2}{||p||(1 + D^{-1})}.$$

Problem 3

(a)

$$\mathbf{y} = \mathbf{Hx} + \mathbf{z}$$

$$\mathbf{Fy} = \mathbf{FSPx} + \mathbf{Fz}$$

$$\Rightarrow \quad \mathbf{Y} = \mathbf{FSF^*DFx} + \mathbf{Fz}$$

$$\Rightarrow \quad \mathbf{Y} = \underbrace{\mathbf{FSF^*D}}_{\mathbf{G}}\mathbf{X} + \mathbf{Z}$$

$$\Rightarrow \quad \mathbf{Y} = \mathbf{GX} + \mathbf{Z}$$

(b)

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q\neq l}\mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \ldots, N-1,$$

Hence,

$$\text{SINR} = \frac{\mathbb{E}\left(|\mathbf{G}_{l,l}\mathbf{X}_l|^2\right)}{\mathbb{E}\left(\left|\sum_{q\neq l}\mathbf{G}_{l,q}\mathbf{X}_q\right|^2\right) + \mathbb{E}\,|Z_l|^2} = \frac{\mathcal{E}_x|\mathbf{G}_{l,l}|^2}{\mathcal{E}_x\sum_{q\neq l}|\mathbf{G}_{l,q}|^2 + \sigma_z^2}$$

(c)

$$\begin{aligned}\mathbb{E}(\mathbf{Y}\mathbf{Y}^*) &= \mathbb{E}((\mathbf{G}\mathbf{X}+\mathbf{Z})(\mathbf{X}^*\mathbf{G}^*+\mathbf{Z}^*)) \\ &= \mathcal{E}_x\mathbf{G}\mathbf{G}^* + \mathbf{I}\sigma_z^2.\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbb{E}(\mathbf{X}_l\mathbf{Y}^*) &= \mathbb{E}(\mathbf{X}_l(\mathbf{X}^*\mathbf{G}^*+\mathbf{Z}^*)) \\ &= \mathbb{1}_l^T\mathcal{E}_x\mathbf{G}^*,\end{aligned} \tag{2}$$

where $\mathbb{1}_l^T = \begin{bmatrix}0 & \cdots & \underbrace{1}_{l^{\text{th}}\text{position}} & 0 & \cdots & 0\end{bmatrix}$. Orthogonality principle implies,

$$\mathbb{E}((\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l)\mathbf{Y}^*) = 0$$

$$\Rightarrow \mathbb{E}(\mathbf{W}_l^*\mathbf{Y}\mathbf{Y}^*) = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)$$

$$\Rightarrow \mathbf{W}_l^* = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)(\mathbb{E}(\mathbf{Y}\mathbf{Y}^*))^{-1}$$

Using equations 1,2 we get that,

$$\Rightarrow \mathbf{W}_l^* = \mathcal{E}_x\mathbb{1}_l^T\mathbf{G}^*(\mathcal{E}_x\mathbf{G}\mathbf{G}^* + \mathbf{I}\sigma_z^2)^{-1}$$

(d)

$$\mathbf{G}_{l,q} = (\mathbf{F}\mathbf{S})_l(\mathbf{F}^*\mathbf{D})_q$$

where $(\mathbf{FS})_l$ denotes the l^{th} row of $\mathbf{FS}$ and $(\mathbf{F}^*\mathbf{D})_q$ denotes the q^{th} column of $\mathbf{F}^*\mathbf{D}$.

$$\mathbf{G}_{l,q} = \frac{1}{N}\begin{bmatrix}e^{j2\pi f_0(N-1)} & e^{j2\pi f_0(N-2)}e^{-j\frac{2\pi}{N}(l-1)} \cdots & e^{j2\pi f_0(N-N)}e^{-j\frac{2\pi}{N}(l-1)(N-1)}\end{bmatrix}\begin{bmatrix}d_q \\ d_q e^{j\frac{2\pi}{N}(q-1)} \\ \vdots \\ d_q e^{j\frac{2\pi}{N}(q-1)(N-1)}\end{bmatrix}$$

$$\Rightarrow \quad \mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\sum_{p=1}^{N}e^{\left(j\frac{2\pi}{N}(q-l)-j2\pi f_0\right)(p-1)}$$

By using the summation formula for the geometric series we get,

$$\mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\left[\frac{1-e^{-j2\pi f_0 N}}{1-e^{j\frac{2\pi}{N}(q-l-f_0N)}}\right] \quad \text{for} \quad f_0 \neq 0.$$

The ICI is significant when $\mathbf{G}_{l,q}$ is comparable to $\mathbf{G}_{l,l}$. When f_0N is large then this could occur, i.e., there is significant time variation over the block.

Problem 4

(a) As seen in section 5.7.2 we have:

$$W(D) = \frac{1}{\gamma_0 ||p|| F^*(D^{-*})} \quad \text{and} \quad B(D) = F(D)$$

(b) The feedback filter is now applied at the transmitter, whereas the feed-forward filter remains on the receiver side. Let's consider notation of the figure 2 without the modulo filter Γ_M. Then we get:

$$\begin{aligned}
\psi_k &= x_k - \sum_{l=1}^{\infty} b_l \psi_{k-l} \\
\sum_{l=0}^{\infty} b_l \psi_{k-l} &= x_k \\
\Psi(D) &= \mathcal{D}(\psi_k) = \frac{X(D)}{B(D)}
\end{aligned}$$

Instead of x_k we transmit ψ_k. After the matched filter the received sequence is:

$$Y(D) = ||p|| Q(D) \Psi(D) + Z(D)$$

Finally we apply the feed-forward filter $W(D)$:

$$\begin{aligned}
Y(D)W(D) &= ||p|| W(D) Q(D) \Psi(D) + Z(D) W(D) \\
&= \frac{||p|| \gamma_0 F(D) F^*(D^{-*})}{\gamma_0 ||p|| F^*(D^{-*})} \frac{X(D)}{F(D)} + Z(D) \frac{1}{\gamma_0 ||p|| F^*(D^{-*})} \\
&= X(D) + \frac{Z(D)}{\gamma_0 ||p|| F^*(D^{-*})}
\end{aligned}$$

ISI has been removed.

(c) The principle about energy boosting and how to prevent it is the same as described in the lecture note (6.1.1). We only need to define properly a modulo function. We can define $\Gamma_M(x_k)$ as a complex modulo reduction of $x_k = a_k + j b_k$ into the interval $[-\frac{M}{2}, \frac{M}{2}] \times [-j\frac{M}{2}, j\frac{M}{2}]$.

$$\Gamma_M(x_k) = x_k - Md \lfloor \frac{a_k + \frac{Md}{2}}{Md} \rfloor - jMd \lfloor \frac{b_k + \frac{Md}{2}}{Md} \rfloor$$

Now, $\tilde{x}_k$ is transmitted in the following way:

$$\begin{aligned}
\tilde{x}_k &= \Gamma_M(\psi_k) \\
&= \Gamma_M[x_k - \sum_{l=1}^{\infty} b_l \tilde{x}_{k-l}]
\end{aligned}$$

At the receiver, after the matched filter we have $R(D) = Y(D)W(D)$ which is equivalent to $r_k = \tilde{x}_k + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k$ in time domain. Doing a modulo operation at the output, we have

$$\begin{aligned}
\Gamma_M(r_k) &= \Gamma_M\left[\tilde{x}_k + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\
&= \Gamma_M\left[\Gamma_M(\psi_k) + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\
&= \Gamma_M\left[\Gamma_M(x_k - \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i}) + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\
&= x_k \oplus_M \Gamma_M(\tilde{z}_k)
\end{aligned}$$