

Homework 3

Problem 1

Consider two individual scalar observations Y_1, Y_2 as,

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= Z_2. \end{aligned} \tag{1}$$

Assume that zero mean Z_1, Z_2 are independent of X and are correlated with covariance,

$$\mathbb{E}[\mathbf{Z}\mathbf{Z}^*] = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho^*\sigma^2 & \sigma^2 \end{bmatrix}, \tag{2}$$

where $\mathbf{Z} = [Z_1, Z_2]^T$. Let $\mathbb{E}[|X|^2] = \mathcal{E}_x$.

(a) Find the best linear MMSE estimate of $\hat{X}$ of the random variable X from the observations Y_1, Y_2.

(b) What is the minimum mean-squared error $\mathbb{E}[|X - \hat{X}|^2]$ of the best linear estimator? Is there a value of ρ for which we get $\mathbb{E}[|X - \hat{X}|^2] = 0$? Interpret the result if $\mathbb{E}[|X - \hat{X}|^2] = 0$ is possible.

(c) Find the best linear estimate $\hat{Z}_1$ of the random variable Z_1 from Y_2. Consider the operation

$$\tilde{Y}_1 = Y_1 - \hat{Z}_1$$

Find the best linear estimate of X from $\tilde{Y}_1$. Is it the same as the answer you found in (a)? Do you have an interpretation?

Problem 2

Let Y_a and Y_b be two separate observations of a zero mean random variable X such that

$$\begin{aligned} Y_a &= H_a X + V_a \\ \text{and} \quad Y_b &= H_b X + V_b, \end{aligned}$$

where $\{V_a, V_b, X\}$ are mutually independent and zero-mean random variables, and $V_a, V_b, X, Y_a, Y_b \in \mathbb{C}$.

(a) Let $\widehat{X}_a$ and $\widehat{X}_b$ denote the linear MMSE estimators for X given Y_a and Y_b respectively. That is

$$\begin{aligned} W_a &= \text{argmin}_{W_a} \mathbb{E}\left(||X - W_a Y_a||^2\right), \\ W_b &= \text{argmin}_{W_b} \mathbb{E}\left(||X - W_b Y_b||^2\right) \end{aligned}$$

and

$$\widehat{X}_a = W_a Y_a \quad \text{and} \quad \widehat{X}_b = W_b Y_b.$$

Find $\widehat{X}_a$ and $\widehat{X}_b$ given that

$$\mathbb{E}\left(XX^*\right) = \sigma_x^2, \mathbb{E}\left(V_a V_a^*\right) = \sigma_a^2, \mathbb{E}\left(V_b V_b^*\right) = \sigma_b^2.$$

Also, find the error variances,

$$\begin{aligned} P_a &= \mathbb{E}\left((X - \widehat{X}_a)(X - \widehat{X}_a)^*\right) \\ P_b &= \mathbb{E}\left((X - \widehat{X}_b)(X - \widehat{X}_b)^*\right) \end{aligned}$$

(b) Prove that

$$P_a^{-1}\widehat{X}_a = \frac{H_a^*}{\sigma_a^2}Y_a, \quad P_b^{-1}\widehat{X}_b = \frac{H_b^*}{\sigma_b^2}Y_b \tag{3}$$

and

$$P_a^{-1} = \frac{1}{\sigma_x^2} + \frac{H_a H_a^*}{\sigma_a^2}, \quad P_b^{-1} = \frac{1}{\sigma_x^2} + \frac{H_b H_b^*}{\sigma_b^2}. \tag{4}$$

Hint: Use the identities

$$\begin{aligned} \mathbf{R}_x\mathbf{H}^*\left[\mathbf{H}\mathbf{R}_x\mathbf{H}^* + \mathbf{R}_v\right]^{-1} &= \left[\mathbf{R}_x^{-1} + \mathbf{H}^*\mathbf{R}_v^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^*\mathbf{R}_v^{-1} \\ \mathbf{R}_x - \mathbf{R}_x\mathbf{H}^*\left[\mathbf{H}\mathbf{R}_x\mathbf{H}^* + \mathbf{R}_v\right]^{-1}\mathbf{H}\mathbf{R}_x &= \left[\mathbf{R}_x^{-1} + \mathbf{H}^*\mathbf{R}_v^{-1}\mathbf{H}\right]^{-1}. \end{aligned}$$

(c) Now we find the estimator $\widehat{X}$, given both observations Y_a and Y_b, i.e.,

$$\begin{pmatrix} Y_a \\ Y_b \end{pmatrix} = \begin{pmatrix} H_a \\ H_b \end{pmatrix} X + \begin{pmatrix} V_a \\ V_b \end{pmatrix}.$$

We want to find the linear MMSE estimate

$$\widehat{X} = \begin{pmatrix} U_a & U_b \end{pmatrix}\begin{pmatrix} Y_a \\ Y_b \end{pmatrix},$$

where

$$\begin{pmatrix} U_a & U_b \end{pmatrix} = \text{argmin}_{(U_a, U_b)}\mathbb{E}\left(||X - \widehat{X}||^2\right)$$

and define the corresponding error variance

$$P = \mathbb{E}\left((X - \widehat{X})(X - \widehat{X})^*\right).$$

Use (3), (4) to show that

$$\begin{aligned} P^{-1}\widehat{X} &= P_a^{-1}\widehat{X}_a + P_b^{-1}\widehat{X}_b \\ \text{and} \quad P^{-1} &= P_a^{-1} + P_b^{-1} - \frac{1}{\sigma_x^2}. \end{aligned}$$