

Homework 2

Problem 1

Consider transmission over an ISI channel and the channel after matched filtering is

$$Y(D) = \| p \| Q(D)X(D) + Z(D).$$

Let $\{q_k\} = < \tilde{\varphi}_0, \tilde{\varphi}_k >$, is given as the following

$$q_k = \begin{cases} 2^{-\frac{|k|-1}{2}} & \text{if } k \text{ is odd,} \\ \frac{5}{3} 2^{-\frac{|k|}{2}} & \text{if } k \text{ is even.} \end{cases}$$

and $S_z(D) = N_0 Q(D)$ and $Q(D)$ is the D-transform of $\{q_k\}$. Find the whitening filter $W(D)$ to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal. That is, $Q(D)W(D)$ is causal.

Problem 2

Consider transmission over an ISI channel with PAM constellation. Let us use

$$\varphi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}(\frac{t}{T})$$

for symbol duration of T and the ISI channel is characterized as $h(t) = \delta(t) - 0.4\delta(t - T)$ and assume that additive white Gaussian noise has power spectral density N_0.

(a) Determine the pulse response $p(t)$.

(b) Find $\| p \|$ and $\tilde{\varphi}(t)$.

(c) Find the autocorrelation function of the noise after sampling the output of the matched filter. Find the whitening filter such that the resulting channel is causal.

(d) Assume that $N_0 = 0.49$, size of PAM is 2, and $x_i \in \{\pm 1\}$. Let the output of the whitened matched filter is $\{0.7, -1.1, -0.2, 0.9, -0.6, 0.9\}$. Find the maximum likelihood sequence using the Viterbi algorithm. Assume the initial and last states are $+1$.

(e) Using the same assumptions as part (d), apply the BCJR algorithm to find the sequence which minimizes the symbol-to-symbol error. Compare the two results and comment.

Problem 3

In this exercise, we will show how the maximum a posteriori (MAP) detection rule can be extended to detection of sequences and also to risk-minimization for sequences.

Let $\mathbf{X}$ be a sequence of length N, where every entry takes values in $\{x_0, x_1\}$. Let $\mathbf{Y}$ be a random sequence (vector) of length N, taking values in $\mathbb{R}^N$. Assume that the statistics of $\mathbf{X}$ are given by $\mathbb{P}[\mathbf{X} = \tilde{\mathbf{x}}] = \prod_{k=0}^{N-1} p(\tilde{x}_k)$ for every possible sequence $\tilde{\mathbf{x}} \in \{x_0, x_1\}^N$. The statistics of $\mathbf{Y}$ are given by some conditional density function $P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ corresponding to an inter-symbol interference channel of memory ν, i.e., $Y(k) = \sum_{i=0}^{\nu} f_i X(k-i) + Z(k)$.

(a) Show that by Bayes rule, the following identity holds

$$\mathbb{P}[X(n) = x_0 | \mathbf{Y} = \mathbf{y}] P_{\mathbf{Y}}(\mathbf{y}) = P_{\mathbf{Y}|X(n)}(\mathbf{y}|x_0)\mathbb{P}[X(n) = x_0].$$

(b) Read the proof of optimality of the MAP rule in Prof. Diggavi's lecture notes. Show that one can carry out the same proof for detecting the symbol $X(n)$ using the whole sequence $\mathbf{y}$. The resulting MAP rule should be

$$\hat{x}(n) = \arg \max_{j \in \{0,1\}} \mathbb{P}[X(n) = x_j | \mathbf{Y} = \mathbf{y}].$$

(c) Now we consider a setup with costs. Let C_{ij} be the cost of detecting x_i when the actual symbol was x_j. Assume that $C_{00} = C_{11} = 0$. Given that we are using the rule $H^{(n)}$ for detecting $X(n)$, the risk function is given by

$$\begin{aligned} R(H^{(n)}) &= C_{01}\mathbb{P}\left[H^{(n)}(\mathbf{Y}) = x_0 | X(n) = x_1\right]\mathbb{P}[X(n) = x_1] \\ &\quad + C_{10}\mathbb{P}\left[H^{(n)}(\mathbf{Y}) = x_1 | X(n) = x_0\right]\mathbb{P}[X(n) = x_0]. \end{aligned}$$

Show that the problem of minimizing this risk is equivalent to minimizing the probability of error of a setup without costs, with the modified prior probabilities:

$$\begin{aligned} \tilde{\mathbb{P}}\big[X(n) = x_1\big] &= \frac{C_{01}\mathbb{P}[X(n) = x_1]}{\alpha} \\ \tilde{\mathbb{P}}\big[X(n) = x_0\big] &= \frac{C_{10}\mathbb{P}[X(n) = x_0]}{\alpha}, \end{aligned}$$

where $\alpha = C_{01}\mathbb{P}[X(n) = x_1] + C_{10}\mathbb{P}[X(n) = x_0]$.

(d) Use everything that you know so far to derive the risk-minimizing MAP rule for detecting $X(n)$.

(e) Write down the basic steps of the BCJR algorithm for risk minimization.