# Homework 1

**Problem 1**[Alarm System]
Consider the following model of an home alarm system. The system has a sensor which checks whether a window has been broken. If a window has been broken the sensor ouputs 1 whereas if no window was broken it outputs 0. Let the set of hypotheses be $\mathcal{H} = \{0, 1\}$, where 0 means no break-in and 1 means break-in. Unfortunately, there is some noise $Z$ added to the sensor output. The noise $Z$ has density

$$f_Z(z) = \begin{cases} 1 - |z|, & |z| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y$ denote the noisy observation. Assume the following priors

$$P_H(0) = \frac{999}{1000}, \qquad P_H(1) = \frac{1}{1000}.$$

Given the sensor output $Y$, you have to decide whether or not you should call the police to alert them to a possible break-in.

a. Starting from first principles, write down the decision rule which minimizes the probability of error. (Hint: Your decision scheme should take the form of a threshold scheme.)

b. The probability of error is not always the best metric. Note that in this example there are two types of error. You can sound an alarm (call the police) but there is no break-in. Call this the probability of *false-alarm*, $p_f$. On the other hand you may not sound an alarm but there is a break-in. Call this the *miss probability*, $p_m$. Assume that in case of a false alarm you have to pay CHF 200 to the police for their effort to come to the house. On the other hand, in case you miss a burglary, assume you loose CHF 10000, because of the stolen items.

   (i) Let $t$ be the threshold of your decision scheme. Write down an expression for the *expected* cost that you incur as a function of $t$. (Hint: Proceed as in question 1. First write down the cost conditioned on $H = 0, 1$.)

(ii) Minimize the expected cost to find the optimum threshold.

**Problem 2**

a. Consider the following channel where $X, N_1, N_2$ are independent *binary* random variables (they take values in $\{0, 1\}$). All additions shown below are modulo two. Equivalently, the additions may be considered as XOR's. We have two hypotheses and the transmitted signals under these hypotheses are $X = 0$ and $X = 1$. We have the following observations:

$$\begin{aligned} Y_1 &= X + N_1, \\ Y_2 &= X + N_2, \\ Y_3 &= X + N_1 + N_2. \end{aligned}$$

(i) Given only $Y_1$, is $Y_3$ relevant?

(ii) Given $Y_1$ and $Y_2$, is $Y_3$ relevant?

b. Consider now the following channel where $X, N_1, N_2$ are independent but real valued. We have two hypotheses and the transmitted signals under these hypotheses are $X_0 = -1$ and $X_1 = 1$. We have the following observations:

$$\begin{aligned} Y_1 &= X + N_1, \\ Y_2 &= X + N_2, \\ Y_3 &= N_1 - N_2. \end{aligned}$$

where $N_1$ and $N_2$ have density functions,

$$p_{N_1}(n) = p_{N_2}(n) = \frac{1}{2}e^{-|n|}.$$

(i) Given $Y_1$, is $Y_2$ relevant?

(ii) Given $Y_1$, is $Y_3$ relevant?

(iii) Given $Y_1$, $Y_2$, is $Y_3$ relevant?

**Problem 3**

Find a set of basis signals for $\cos(t), \cos(t + \pi/3), \cos(t + \pi/2)$.

**Problem 4**

a. Let $s(t)$ be a real signal in $\mathcal{L}^2$ that is bandlimited to $[B_1, B_2]$, i.e., $S(f) = 0$ for $|f| \notin [B_1, B_2]$. We would like to sample this function with a sampling period

$T$ and later reconstruct it perfectly. What is the largest sampling period $T$? To justify your answer, use the sampling theorem and outline all additional steps required.

Hint: Do we need to sample at $T = \frac{1}{2B_2}$ ?

b. Having the samples $\{s_{nT}\}_{n=-\infty}^{\infty}$ from part (a), we now want to reconstruct the signal $s(t)$ perfectly. Recall that we could use the sinc function for reconstruction as this pulse satisfies the interpolation formula

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)\mathrm{sinc}(\frac{t}{T} - n)$$

where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. However, we are asked to use the real valued function $q(t)$, whose Fourier transform $Q(f)$ satisfies

$$|Q(f)| = \begin{cases} \sqrt{T - T^2 f} & 0 \leq f \leq \frac{1}{T} \\ 0 & \frac{1}{T} < f \end{cases}.$$

Is it possible to reconstruct the signal by only replacing the sinc function with $q(t)$ in the interpolation formula given above? That is, is it true that

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)q(\frac{t}{T} - n) \qquad ?$$

Hint: Use the Nyquist criterion.