ON THE EQUALITY OF EDGE AND BULK CONDUCTANCE IN THE INTEGER HALL EFFECT: MICROSCOPIC ANALYSIS

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Abstract. The bulk Hall conductance can be computed as a response function of an infinite system, while the edge conductance can be found from the total edge current for a half-infinite system. From general heuristic arguments one expects that these two quantities are equal, and indeed this has been proven recently for the integer effect. Here we provide a new elementary self-contained proof of this fact using the notion of relative index between two projectors.

1. Introduction and Results.

In a finite Hall sample Λ, the linear response law for the bulk current density in a Hall state with vanishing longitudinal conductance is $j_B = -\sigma_H \chi_\Lambda \hat{z} \wedge E$, where χ_Λ is the characteristic function of Λ, $E = -\nabla\Phi$ the electric field in the sample and σ_H the Hall conductance. Since $\nabla \cdot j_B = -\sigma_H \delta_{\partial\Lambda} t \cdot E$, where t is the tangent vector to the boundary $\partial\Lambda$, there must exist an edge current density j_E such that $\nabla \cdot (j_B + j_E) = 0$,

$$\nabla \cdot j_E = \sigma_H \delta_{\partial\Lambda} t \cdot E \tag{1.1}$$

According to the analysis of Halperin [1] the edge current is concentrated within a few magnetic lengths from the boundary and flows along it, without dissipation, between two reservoirs connected to the sample at points r_1 and r_2 on $\partial\Lambda$. The total edge current flowing through a section perpendicular to $\partial\Lambda$ is given by

$$I_E = \sigma_E(\Phi(r_1) - \Phi(r_2)) \tag{1.2}$$

where $E = -\nabla\Phi(r)$. Here it is implicitely assumed that the chemical potential difference between the two reservoirs is created only by the electric field. Let Γ be a section perpendicular to $\partial\Lambda$ between the points r_1 and r_2. Since I_E and the edge current density are related by $I_E = \int_\Gamma t \cdot j_E$, by integrating (1.1) and comparing with (1.2) we find that necessarily $\sigma_H = \sigma_E$.

The equality between edge and bulk conductance, deduced above from heuristic considerations, has been rigorously proven recently, in a microscopic setting in the case of the integer Hall effect [2], [3]. In [2] techniques from K-theory and non-commutative geometry are used, while [3] is based on a generalisation of the notion of relative index between two projectors. In the present work we give a new elementary proof which uses only the usual notion of relative index between two projectors, a form of Fredholm index introduced by Avron, Seiler and Simon [4]. We believe that our analysis is simpler than the previous ones and is more closely related to the early argument of charge transport of Laughlin [5]. Although it is clear that the heuristic argument applies also to the fractional Hall effect, the microscopic treatments are based on one particle hamiltonians and do not cover this situation. There the equality of bulk and edge conductance can also be established by general considerations of gauge invariance and anomaly cancellation in a Chern-Simons field theory approach [6], [7].

All our hamiltonians are defined on a square lattice and act on the space $l^2(\mathbf{Z^2})$. The bulk hamiltonian of the infinite system has matrix elements

$$\langle r|H_B|r'\rangle = t_{r,r'}e^{i\phi_{r,r'}} + U_r\delta_{r,r'} \tag{1.3}$$

where the kinetic energy is given by a nearest neighbour hopping term $t_{r,r'} = t_0 \geq 0$ for $|r-r'| = 1$, and vanishes otherwise. The phase factors satisfy $\phi_{r,r'} = -\phi_{r',r}$, and describe the effect of a magnetic flux through each plaquette. We do not require the flux to be uniform. In the potential term U_r are real, uniformly bounded, $\sup_r |U_r| = v$, but otherwise arbitrary. More general tight binding models could be considered but we limit ourselves to (1.3) for simplicity. The spectrum of H_B is contained in the interval $[-2t_0 - v, 2t_0 + v]$ and may be a very complicated object to study. However a detailed knowledge of its nature is not needed for our analysis which only requires the

Assumption: The spectrum of H_B has at least one gap G.

This assumption is fulfilled, for example, in the Hofstadter model corresponding to the case of uniform flux (rational) and $v = 0$. Clearly this is still the case if $0 < v << t_0$. We expect our main result to still hold if this assumption is replaced by the physically important case where there is a "mobility gap" instead of a spectral gap.

Let P_F be the Fermi projector on the interval $]-\infty, E_F]$. The Kubo formula for the bulk conductance takes the form

$$\sigma_H = -\lim_{\epsilon\to 0}\lim_{\Lambda\uparrow\mathbf{Z^2}}\frac{1}{|\Lambda|}\sum_{r\in\Lambda}\int_{-\infty}^{0} dte^{\epsilon t}\langle r|e^{-itH_B}[H_B, y]e^{itH_B}[x, P_F]|r\rangle \tag{1.4}$$

where $(x, y) = r$ and the thermodynamic limit is taken along a sequence of squares $\Lambda \subset \mathbf{Z^2}$. For a periodic potential it was recognised by Thouless et al [8] that, when $E_F \in G$, (1.4) is a Chern number associated to the fibre bundle over the Brillouin zone. Even if the potential has no periodicity, (1.4) can still be written in the form of a "non-commutative" Chern number* [9], [10], [11] as long as E_F lies in a mobility gap (in particular if $E_F \in G$)

$$\sigma_H = 2\pi i \lim_{\Lambda\uparrow\mathbf{Z^2}}\frac{1}{|\Lambda|}\sum_{r\in\Lambda}\langle r|P_F[[x, P_F], [y, P_F]]|r\rangle \tag{1.5}$$

* in [9] this formula appears in a continuum space setting

Let us now define the edge conductance of a semi-infinite system $\mathbf{Z} \times \mathbf{N}$ with one edge. The hamiltonian of the semi-infinite system is equal to

$$\langle r|H_E|r'\rangle = \langle r|H_B|r'\rangle \qquad \text{for} \qquad y \geq 0 \qquad \text{and} \qquad y' \geq 0, \tag{1.6}$$

and vanishes otherwise. Let P_Δ be the spectral projector of H_E on an open interval $\Delta \subset G$ and containing E_F. It is natural to define the edge conductance as the total current carried in Δ, across a section perpendicular to the edge, divided by $|\Delta|$,

$$\sigma_E = -\frac{1}{2i|\Delta|} \sum_{y \geq 0} \langle r|[x, H_E]P_\Delta + P_\Delta[x, H_E]|r\rangle \tag{1.7}$$

Remarks:

(i) In appendix A we show that for $\Delta \subset G$, the sum is absolutely convergent. The reason is that basically as $y \to +\infty$ P_Δ can be replaced by the spectral projector of H_B on Δ which vanishes. For Δ in a region of localised states the sum is presumably not absolutely convergent and a cutoff procedure would be needed.

(ii) In [2] a limit $\lim_{\Delta \to 0}$ is used. We can also cover this situation, but our result holds for a finite interval which is physically more satisfying. In [3] a regularised definition is used, but it will become clear that it is equivalent to ours.

(iii) It will become clear later on that (1.7) is independent of $x \in \mathbf{Z}$.

Our main result is

Theorem. For any open interval $\Delta \subset G$ where the inclusion is strict and any $E_F \in \Delta$ we have

$$\sigma_E = \sigma_H$$

As already alluded to above, we expect that the same holds for almost every realisation of a random potential U_r, when Δ lies in a region of localised states and $E_F \in \Delta$. Besides the question of the definition (1.7) the main problem with our approach is that we use an adiabatic theorem which forces us to take $\Delta \subset G$. But we point out that we need only o(1) estimates in the time scale and that there are gapless cases where such weak adiabatic theorems are known to hold (see [12] for a recent review of the situation). It would also be desirable to prove such a result in a continuum space setting. In fact, using the results in [13] we are able to do this, up to a proof of lemma 5 (see next section).

2. Main ideas of the proof.

An essential feature of our strategy is to approximate σ_E by the conductance of a system defined on the complement of an open disc $D_{x,R}$ of integer radius R and centered at $(x+\frac{1}{2},-\frac{1}{2}-R)$, $x \in \mathbf{Z}$. We also pierce the center of the disc with a flux line Φ. Let (r,r') be a directed nearest neighbour bond with the direction chosen such that $x' < x$ or $y' > y$. We denote by $\alpha_{r,r'}$ the angle of sight from the center of the disc, of the directed bond (r,r'); note that $\alpha_{r,r'}$ tends to zero as r and r' tend to infinity. In the "symmetric gauge" the corresponding hamiltonian is

$$\langle r|H_R(\Phi)|r'\rangle = t_{r,r'}e^{i\phi_{r,r'}-i\frac{\Phi}{2\pi}\alpha_{r,r'}} + U_r\delta_{r,r'} \qquad \text{for} \qquad r,r' \in D^c_{x,R} \tag{2.1}$$

and is equal to zero if r or r' belong to $D_{x,R}$. Note that for $R=0$, $H_0(\Phi)$ is nothing else than the bulk hamiltonian with an extra flux Φ through the plaquette centered around the point $(x+\frac{1}{2},-\frac{1}{2})$.

The essential spectrum of $H_R(\Phi)$ is the same as the one of H_B. Indeed $H_R(\Phi) = H_R(0) + K(\Phi)$ with

$$\langle r|K(\Phi)|r'\rangle = t_{r,r'}e^{i\phi_{r,r'}}(e^{-i\frac{\Phi}{2\pi}\alpha_{r,r'}} - 1) \qquad \text{for} \qquad r,r' \in D^c_{x,R} \tag{2.2}$$

and zero otherwise. It is clear that $H_B - H_R(0)$ is finite rank, so it is enough to show that $K(\Phi)$ is compact. A direct computation of the matrix elements of $(K(\Phi))^2$ and the bound $|\langle r|K(\Phi)|r'\rangle| \leq t_0\delta_{|r-r'|,1}|r|^{-1}$ imply that $(K(\Phi))^2$ is Hilbert-Schmidt. Therefore $(K(\Phi))^4$ is trace-class which implies that $K(\Phi)$ is compact. Thus for any $\Delta \subset G$ (strict inclusion), $\Sigma(H_R(\Phi)) \cap \Delta$ contains only discrete isolated eigenvalues which form analytic branches labelled $E_l(\Phi)$ as Φ varies from 0 to 2π.

Let $g \in C^\infty(\mathbf{R})$ be such that $g(\lambda) = 1$ for $\lambda \leq \inf\Delta$, $g(\lambda) = 0$ for $\lambda \geq \sup\Delta$, and $-\frac{2}{|\Delta|} < g'(\lambda) < 0$ for $\lambda \in \Delta$. We define the edge conductance associated to $D^c_{x,R}$ as follows

$$\sigma_R = \int_0^{2\pi} \frac{d\Phi}{2\pi}\sigma_R(\Phi) \tag{2.3}$$

$$\sigma_R(\Phi) = -\sum_{E_l(\Phi)\in\Delta} \frac{\partial}{\partial\Phi} g(E_l(\Phi)) \tag{2.4}$$

In appendix A we prove a technical Lemma

Lemma 1. For any function g defined as above we have

$$\lim_{R\to+\infty} \sigma_R = \sigma_E \tag{2.5}$$

We consider an interval $\tilde{G}$, such that $\Delta \subset \tilde{G} \subset G$ and $|\Delta| < |\tilde{G}| < |G|$. We assume in the following that there are no crossings between branches in $\tilde{G}$. This is not a loss of generality because if there are crossings we add a small finite rank perturbation to $H_R(\Phi)$

which lifts all degeneracies. In appendix B we construct this perturbation explicitly and explain why it does not change all the subsequent analysis. We cannot take $\tilde{G} = G$ because we cannot rule out the existence of an accumulation point of eigenvalues at the boundary of G. Now we consider the spectral flow for the branches that lie entirely in $\tilde{G}$. Since $H_R(2\pi)$ and $H_R(0)$ are unitarily equivalent and the branches $E_l(\Phi)$ do not cross, the spectral flow satisfies

$$E_l(2\pi) = E_{l+n}(0) \tag{2.6}$$

It is important to remark that in (2.3) the integer n (which we assume positive; if it is negative all the arguments can be adapted) is equal to the number of branches that cross the Fermi energy E_F an odd number of times. In fact it follows from the lemmas 3 and 4 below that n is independent of R.

The main core of the proof of the theorem is formed by the four lemmas below.

Lemma 2. Let n be the integer in (2.6). Then for R large enough we have

$$\sigma_R = n \tag{2.7}$$

We relate the integer n to the relative index between two projectors. Let $P_R(\Phi)$ be the Fermi projector of $H_R(\Phi)$ on the interval $]-\infty, E_F]$ and U the unitary operator $U|r\rangle = e^{iarg(r)}|r\rangle$. One has $UP_R(0)U^\dagger = P_R(2\pi)$. We introduce a time dependent Hamiltonian $H_R(\phi_T(s))$ where $\phi_T(s)$, $0 \leq s \leq T$ is a smooth function describing the slow switching of a flux quantum: $\phi_T(0) = 0$, $\phi_T(T) = 2\pi$. The associated unitary evolution operator is $U_{T,R}(s) = \mathrm{T}[\exp -i \int_0^s ds' H_R(\phi_T(s'))]$.

Lemma 3. Given R, one can find T large enough such that n is given by the following relative index

$$n = \mathrm{Ind}\big(U_{T,R}(T)P_R(0)U_{T,R}^\dagger(T); UP_R(0)U^\dagger\big) \tag{2.8}$$

This formula is in fact a mathematical expression of Laughlin's gedanken experiment. Indeed formally the relative index is (although this trace is not defined)

$$\mathrm{Tr}\big(U_{T,R}(T)P_R(0)U_{T,R}^\dagger(T) - UP_R(0)U^\dagger\big)$$

which can be interpreted as minus the number of electrons transported outside of the system in the process of switching a flux quantum adiabatically. Note that this formula closely resembles the one introduced in [4] but it is different because in [4] the time evolution operator is absent. Using stability properties of the relative index under compact perturbations we will show that one can restore the disc $D_{x,R}$ without changing the index. We set $U_{T,R=0}(s) = U_T(s)$ and recall $P_{R=0}(0) = P_F$.

Lemma 4.

$$\mathrm{Ind}\big(U_{T,R}(T)P_R(0)U_{T,R}^\dagger(T); UP_R(0)U^\dagger\big) = \mathrm{Ind}\big(U_T(T)P_FU_T^\dagger(T); UP_FU^\dagger\big) \tag{2.9}$$

From the work of Avron, Seiler, Simon [4], [11] we know that when E_F lies in a gap (or a region of localised states) of H_B the bulk Hall conductance (1.5) satisfies

$$\sigma_H = \mathrm{Ind}\big(P_F; UP_FU^\dagger\big) \tag{2.10}$$

The equality $\sigma_E = \sigma_H$ then follows from lemmas 1, 2, 3, 4 and

Lemma 5.

$$\mathrm{Ind}\big(U_T(T)P_FU_T^\dagger(T); UP_FU^\dagger\big) = \mathrm{Ind}\big(P_F; UP_FU^\dagger\big) \tag{2.11}$$

In the next section we prove Lemmas 2, 3, 4 and 5. Appendices A and B contain more technical material. We denote by C positive numerical constants.

3. Proof of Lemmas 2, 3, 4 and 5

Proof of Lemma 2.

We assume that n is positive; for n negative the argument is similar. We take R large enough so that all branches entering in the argument lie entirely within $\tilde{G}$ so that (2.6) holds. Let l_- (resp l_+) be the largest (resp smallest) index such that $E_{l_-}(2\pi)$ (resp $E_{l_+}(0)$) lies below (resp above) Δ. Thus $l_- + 1, ..., l_- + n$ (resp $l_+ - n, ..., l_+ - 1$) cross $\inf \Delta$ (resp $\sup \Delta$) an odd number of times and $l_- + n + 1, ..., l_+ - n - 1$ either remain entirely in Δ or cross $\inf \Delta$, $\sup \Delta$ an even number of times. Since $\frac{\partial}{\partial \Phi} g(E_l(\Phi)) = 0$ for $E_l(\Phi) \notin \Delta$

$$\sigma_R = -\sum_{l_-+1}^{l_+-1} \int_0^{2\pi} \frac{\partial}{\partial \Phi} g(E_l(\Phi)) \frac{d\Phi}{2\pi} \tag{3.1}$$

and from (2.3)

$$\sigma_R = -\sum_{l_-+1}^{l_+-1} \big(g(E_{l+n}(0)) - g(E_l(0))\big) = \sum_{i=1}^{n} g(E_{l_-+i}(0)) - \sum_{i=1}^{n} g(E_{l_+-1+i}(0)) \tag{3.2}$$

One remarks that $g(E_{l_-+i}(0)) = 1$ and $g(E_{l_+-1+i}(0)) = 0$ for $i = 1, ..., n$, which proves (2.7).

For the convenience of the reader we summarize a few useful results developed in [4]. Let P and Q be orthogonal projections on a separable Hilbert space $\mathcal{H}$. The pair $(P, Q)$is called a Fredholm pair if $QP : P\mathcal{H} \to Q\mathcal{H}$ is a Fredholm operator. The "relative index" $\mathrm{Ind}(P; Q)$ of the pair, is the usual Fredholm index of the map $T = QP$, that is, $\mathrm{dimker} T^\dagger T - \mathrm{dimker} TT^\dagger$. One proves that (P, Q) is a Fredholm pair if and only if 1 and -1 are isolated finitely degenerate eigenvalues of $P - Q$. Moreover one has $\mathrm{Ind}(P; Q) = \mathrm{dimker}(P - Q - 1) - \mathrm{dimker}(P - Q + 1)$. A useful formula (we use it below for $m = 0$ only) states that if $(P - Q)^{2m+1}$ is trace-class for some integer m, then $(P; Q)$ is Fredholm pair and $\mathrm{Ind}(P; Q) = \mathrm{Tr}(P - Q)^{2n+1}$ for all $n \geq m$ A central result on which we rely is that if

$(P;Q)$ and $(Q;R)$ are Fredholm pairs and either $P-Q$ or $Q-R$ is compact then $(P;R)$ is also a Fredholm pair and

$$\mathrm{Ind}(P;R) = \mathrm{Ind}(P;Q) + \mathrm{Ind}(Q;R) \tag{3.3}$$

Finaly we note that if $(P;Q)$ is Fredholm then so is $(UPU^\dagger;UQU^\dagger)$ for any unitary U and the relative index remains invariant. Also $\mathrm{Ind}(P;Q) = -\mathrm{Ind}(Q;P)$.

Proof of Lemma 3

Again we assume n positive. Let $\mathcal{L}_c$ the set of levels $E_l(0)$ of $H_R(0)$ such that the branch $E_l(\Phi)$ crosses E_F an odd number of times. Clearly the set $\mathcal{L}_c$ is finite since the spectrum has no accumulation points in Δ, and is non empty for R large enough since $\max |E_l(\Phi) - E_l(\Phi')| \leq \frac{2t_0}{R}$. Let $P_R^c(0)$ be the eigenprojector of $H_R(0)$ on $\mathcal{L}_c$, and $P_R^{n.c}(0) = P_R(0) - P_R^c(0)$. $P_R^{n.c}(0)$ is the eigenprojector of $H_R(0)$ over the levels $E_l(0) < E_F$ corresponding to branches that either do not cross the Fermi level or cross it an even number of times. We have

$$n = \mathrm{Tr} P_R^c(0) = \mathrm{Tr}(P_R(0) - P_R^{n.c}(0)) = \mathrm{Ind}(P_R(0), P_R^{n.c}(0)) \tag{3.4}$$

where the pair $(P_R(0); P_R^{n.c}(0))$ is Fredholm because the difference of projectors is finite rank.

Consider now the evolved projector $U_{T,R}(T) P_R^{n.c}(0) U_{T,R}^\dagger(T)$. Since the branches $E_l(\Phi)$ do not cross, the adiabatic theorem tells us that as $T \to \infty$ the evolved projector tends to the Fermi projector $P_R(2\pi)$. More precisely, given R, one can find T large enough so that

$$||U_{T,R}(T) P_R^{n.c}(0) U_{T,R}^\dagger(T) - P_R(2\pi)|| < 1 \tag{3.5}$$

Thus $\big(U_{T,R}(T) P_R^{n.c}(0) U_{T,R}^\dagger(T); P_R(2\pi)\big)$ is a Fredholm pair and

$$\mathrm{Ind}\big(U_{T,R}(T) P_R^{n.c}(0) U_{T,R}^\dagger(T); P_R(2\pi)\big) = 0 \tag{3.6}$$

Now $\big(U_{T,R}(T) P_R(0) U_{T,R}^\dagger(T); U_{T,R}(T) P_R^{n.c}(0) U_{T,R}^\dagger(T)\big)$ is also Fredholm and has finite rank difference, thus from (3.3), (3.4) and (3.5) we find

$$n = \mathrm{Ind}\big(U_{T,R}(T) P_R(0) U_{T,R}^\dagger(T); P_R(2\pi)\big) = \mathrm{Ind}\big(U_{T,R}(T) P_R(0) U_{T,R}^\dagger(T); U P_R(0) U^\dagger\big) \tag{3.7}$$

Proof of Lemma 4.

First we remark that $(P_R(0), P_F)$ has a compact difference and is therefore Fredholm. Indeed $H_R(0) - H_B$ is finite rank so that (Γ the square contour going through the corners $E_F + i$, $E_F - i$, $E_0 - i$, $E_0 + i$ with $E_0 < -2t_0 - v$)

$$P_R(0) - P_F = \int_\Gamma dz \frac{1}{z - H_R(0)} (H_R(0) - H_B) \frac{1}{z - H_B} \tag{3.8}$$

has finite Hilbert-Schmidt norm. Then, applying (3.3)

$$\begin{aligned}
&\mathrm{Ind}\big(U_{T,R}(T)P_R(0)U^\dagger_{T,R}(T);UP_R(0)U^\dagger\big)\\
&=\mathrm{Ind}\big(U_{T,R}(T)P_R(0)U^\dagger_{T,R}(T);UP_FU^\dagger\big)+\mathrm{Ind}\big(UP_FU^\dagger(T);UP_R(0)U^\dagger\big)\\
&=\mathrm{Ind}\big(U_{T,R}(T)P_R(0)U^\dagger_{T,R}(T);U_{T,R}(T)P_FU^\dagger_{T,R}(T)\big)+\mathrm{Ind}\big(U_{T,R}(T)P_FU^\dagger_{T,R}(T);UP_FU^\dagger\big)\\
&+\mathrm{Ind}\big(UP_FU^\dagger;UP_R(0)U^\dagger\big)
\end{aligned} \tag{3.9}$$

The first and third terms in the last member of (3.9) cancel, while the second one can be rewritten as a usual fredholm index

$$\mathrm{Ind}\big(U_{T,R}(T)P_FU^\dagger_{T,R}(T);UP_FU^\dagger\big)=\mathrm{Ind}\big(P_FU^\dagger U_{T,R}(T)P_F|P_F\mathcal{H}\to P_F\mathcal{H}\big) \tag{3.10}$$

We recall the notation $U_T(s)=U_{T,R=0}(s)$. We have

$$P_FU^\dagger U_{T,R}(T)P_F=P_FU^\dagger U_T(T)P_F+P_FU^\dagger(U_{T,R}(T)-U_T(T))P_F \tag{3.11}$$

and from the formula

$$U_{T,R}(T)-U_T(T)=\int_0^T dsU_T(T-s)(H_R(0)-H_B)U_{T,R}(s) \tag{3.12}$$

we see that the second term on the right hand side of (3.11) has a finite Hilbert-Schmidt norm (because $H_R(0)-H_B$ is finite rank). Thus by the stability of the usual Fredholm index under compact perturbations we can replace $U_{T,R}(T)$ by $U_T(T)$ in (3.10). Combining this with (3.9) we obtain

$$\begin{aligned}
\mathrm{Ind}\big(U_{T,R}(T)P_R(0)U^\dagger_{T,R}(T);UP_R(0)U^\dagger\big)&=\mathrm{Ind}\big(P_FU^\dagger U_T(T)P_F|P_F\mathcal{H}\to P_F\mathcal{H}\big)\\
&=\mathrm{Ind}\big(U_T(T)P_FU^\dagger_T(T);UP_FU^\dagger\big)
\end{aligned} \tag{3.13}$$

Proof of Lemma 5

First we show that it is enough to establish that the difference $U_T(s)P_FU^\dagger_T(s)-P_F$ is compact for all s. If this is the case,

$$\begin{aligned}
U_T(s)P_FU^\dagger_T(s)-U_T(s-\epsilon)P_FU^\dagger_T(s-\epsilon)&=\big(U_T(s)P_FU^\dagger_T(s)-P_F\big)\\
&-\big(U_T(s-\epsilon)P_FU^\dagger_T(s-\epsilon)-P_F\big)
\end{aligned} \tag{3.14}$$

is also compact for all s, and by continuity of the time evolution in the operator norm (seen by integrating the Schrödinger equation over the time interval $[s,s-\epsilon]$)

$$\mathrm{Ind}\big(U_T(s)P_FU^\dagger_T(s);U_T(s-\epsilon)P_FU^\dagger_T(s-\epsilon)\big)=0 \tag{3.15}$$

for $\epsilon>0$ small enough. Decomposing the time interval $[0,T]$ in a finite number of small time intervals and applying (3.3) we get

$$\mathrm{Ind}\big(U_T(T)P_FU^\dagger_T(T);P_F\big)=0 \tag{3.16}$$

The result of the Lemma follows by applying again (3.3) to the pairs $\left(U_T(T)P_FU_T^\dagger(T); P_F\right)$ and $\left(P_F; UP_FU^\dagger\right)$.

Now integrating the Heisenberg equation of motion we have

$$U_T(s)P_FU_T^\dagger(s) - P_F = i\int_0^s ds_1[H_0(\phi_T(s_1)), U_T(s_1)P_FU_T^\dagger(s_1)] \tag{3.17}$$

Iterating (3.17) we obtain the perturbation series

$$\begin{aligned} &U_T(s)P_FU_T^\dagger(s) - P_F \\ &= \sum_{k=1}^{\infty} i^k \int_0^s ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{k-1}} ds_k [H_0((\phi_T(s_1)), [H_0((\phi_T(s_2)), [...[H_0((\phi_T(s_k)), P_F]...]]] \end{aligned} \tag{3.18}$$

Since $H_0(\Phi) = H_B + K(\Phi)$ with $K(\Phi)$ given by (2.2) (with $R = 0$), $[H_0((\phi_T(s)), P_F] = [K((\phi_T(s)), P_F]$. This commutator has a kernel bounded by

$$\begin{aligned} &\sum_{r''}\left(t_{r,r''}|e^{i\phi_T(s)\alpha_{r,r''}} - 1||\langle r''|P_F|r'\rangle| + t_{r,r''}|\langle r|P_F|r''\rangle||e^{i\phi_T(s)\alpha_{r'',r'}} - 1|\right) \\ &\le \frac{C}{|r|}|\langle r|P_F|r'\rangle| \le \frac{C}{|r|}e^{-C|r-r'|} \end{aligned} \tag{3.19}$$

where the last inequality comes from the fact that E_F is in a gap of H_B. Using the following estimate for the norm of the p-th trace ideal of compact operators (see [11])

$$||A||_p \le \sum_{r'}\left(\sum_r |\langle r + r'|A|r\rangle|^p\right)^{\frac{1}{p}} \tag{3.20}$$

we conclude from (3.19) that

$$||[H_0((\phi_T(s)), P_F]||_3 \le C_3 \tag{3.21}$$

where C_3 is a numerical constant independent of s. Using (3.21), (3.18) and $||H_0(\Phi)|| \le 2t_0 + v$ and $||BAC||_p \le ||B||||A||_p||C||$ we find

$$||U_T(s)P_FU_T^\dagger(s) - P_F||_3 \le C_3\sum_{k=1}^{\infty}\frac{s^k}{k!}2^{k-1}(2t_0 + v)^{k-1} \tag{3.22}$$

a convergent sum for all s. In particular, $(U_T(s)P_FU_T^\dagger(s) - P_F)$ is a compact operator.

Appendix A.

The proof of Lemma 1 is based on two auxiliary results.

Lemma A.1. There exist a positive constant $C > 0$ such that for $\Delta \subset G$ and $y \geq 0$,

$$|\langle x, y|P_\Delta|x, y\rangle| \leq Ce^{-Cy} \tag{A.1}$$

Lemma A.2. For any integer $N > 4$, there exists $C_N > 0$ independent of R such that for all $x \in \mathbf{Z}$ and $y \geq 0$, we have

$$|\langle x \pm 1, y|g'(H_R(\Phi)|x, y\rangle - \langle x \pm 1, y|g'(H_E)|x, y\rangle| \leq \frac{C_N}{(\sqrt{R} + y)^{N-4}} \tag{A.2}$$

Proof of Lemma A.1

By Stone's formula, for $\Delta \subset G$

$$\begin{aligned} |\langle x, y|P_\Delta|x, y\rangle| &= \lim_{\epsilon\to 0}\int_\Delta du\Big(\langle x, y|\frac{1}{u + i\epsilon - H_E}|x, y\rangle - \langle x, y|\frac{1}{u - i\epsilon - H_E}|x, y\rangle\Big) \\ &= \lim_{\epsilon\to 0}\int_\Delta du\Big(\langle x, y|\frac{1}{u + i\epsilon - H_E} - \frac{1}{u + i\epsilon - H_B}|x, y\rangle \\ &- \langle x, y|\frac{1}{u - i\epsilon - H_E} - \frac{1}{u - i\epsilon - H_B}|x, y\rangle\Big) \end{aligned} \tag{A.3}$$

Thanks to the resolvent identity (A.3) can be transformed into

$$\begin{aligned} |\langle x, y|P_\Delta|x, y\rangle| &= \lim_{\epsilon\to 0}\int_\Delta du\langle x, y|\frac{-2i\epsilon}{(u - H_E)^2 + \epsilon^2}(H_B - H_E)\frac{1}{u - i\epsilon - H_B}|x, y\rangle \\ &- \lim_{\epsilon\to 0}\int_\Delta du\langle x, y|\frac{1}{u - i\epsilon - H_E}(H_B - H_E)\frac{2i\epsilon}{(u - H_B)^2 + \epsilon^2}|x, y\rangle \\ &= \lim_{\epsilon\to 0} I_1(\epsilon) - \lim_{\epsilon\to 0} I_2(\epsilon) \end{aligned} \tag{A.4}$$

The first term in (A.3) is

$$\begin{aligned} I_1(\epsilon) =& 2t_0 i\int_\Delta du\sum_{x'}\langle x, y|\frac{\epsilon}{(u - H_E)^2 + \epsilon^2}|x', 0\rangle \\ &\times \big(\langle x' - 1, -1| + \langle x' + 1, -1|\big)\frac{1}{u + i\epsilon - H_B}|x, y\rangle \end{aligned} \tag{A.5}$$

Applying the Combes-Thomas estimate,

$$|\langle x' \pm 1, -1|\frac{1}{u + i\epsilon - H_B}|x, y\rangle| \leq Ce^{-C(y + |x - x'|)} \tag{A.6}$$

then the Cauchy-Schwartz inequality,

$$\int_\Delta du|\langle x,y|\frac{\epsilon}{(u-H_E)^2+\epsilon^2}|x',0\rangle| \le \Big(\int_\Delta du|\langle x,y|\frac{\epsilon}{(u-H_E)^2+\epsilon^2}|x,y\rangle|\Big)^{1/2} \Big(\int_\Delta du|\langle x',0|\frac{\epsilon}{(u-H_E)^2+\epsilon^2}|x',0\rangle|\Big)^{1/2} \tag{A.7}$$

and then the spectral decomposition,

$$\begin{aligned}\int_\Delta du|\langle x,y|\frac{\epsilon}{(u-H_E)^2+\epsilon^2}|x,y\rangle &= \int_\Delta du\int d\lambda\frac{\epsilon}{(u-\lambda)^2+\epsilon^2}\langle x,y|dE(\lambda)|x,y\rangle \\ &\le \frac{1}{\pi}\int d\lambda\langle x,y|dE(\lambda)|x,y\rangle = \frac{1}{\pi}\end{aligned} \tag{A.8}$$

(obtained by extending the u-integral to $\mathbf{R}$) we get

$$I_1(\epsilon) \le \frac{4t_0}{\pi}e^{-Cy}\sum_{x'}e^{-C|x-x'|} \le Ce^{-Cy} \tag{A.9}$$

uniformly in ϵ.

The second term in (A.4) is

$$\begin{aligned}I_2(\epsilon) = 2t_0 i\int_\Delta du\sum_{x'}\langle x,y|\frac{1}{u+i\epsilon-H_E}|x',0\rangle \\ \times\big(\langle x'-1,0|+\langle x'+1,0|\big)\frac{\epsilon}{(u-H_B)^2+\epsilon^2}|x,y\rangle\end{aligned} \tag{A.10}$$

From

$$|\langle x,y|\frac{1}{u+i\epsilon-H_E}|x',0\rangle| \le \frac{1}{\epsilon} \tag{A.11}$$

and Cauchy's formula with $dist(\Delta,\Gamma)>1$ together with (A.6)

$$\begin{aligned}|\langle x'\pm 1,0|\frac{\epsilon}{(u-H_B)^2+\epsilon^2}|x,y\rangle| &\le \int_\Gamma |dz|\frac{\epsilon}{(u-z)^2+\epsilon^2}|\langle x'\pm 1,0|\frac{1}{z-H_B}|x,y\rangle| \\ &\le C\epsilon e^{-C(y+|x-x'|)}\end{aligned} \tag{A.12}$$

we obtain

$$|I_2(\epsilon)| \le Ce^{-Cy} \tag{A.13}$$

uniformly in ϵ. The claim of the Lemma follows from (A.13), (A.9) and (A.3).

Proof of Lemma A.2.

We use the Helfer-Sjostrand formula in the following set-up. Let z be a complex number and u the real part, w the imaginary part. Set $\partial_{\bar{z}} = \partial_u + i\partial_w$ and let $\tilde{g'}$ a quasi-analytic extension of g', say

$$\tilde{g'}(z) = \sum_{k=0}^{N} g^{k+1}(u)\frac{(iw)^k}{k!}\chi(w) \tag{A.14}$$

with $N > 4$ and $\chi \in C_0^\infty$, even and equal to 1 for $y \in]-\delta, \delta[$, $\delta > 0$. We have

$$\partial_{\bar{z}}\tilde{g'}(z) = g^{N+2}(u)\frac{(iw)^N}{N!}\chi(w) + i\sum_{k=0}^{N} g^{k+1}(u)\frac{(iw)^k}{k!}\chi'(w) \tag{A.15}$$

Thus $\partial_{\bar{z}}\tilde{g'}(z)$ has a compact support S in the complex plane and

$$|\partial_{\bar{z}}\tilde{g'}(z)| \le C|w|^N \tag{A.16}$$

for some numerical constant $C > 0$ (depending on δ and N). The Helfer-Sjostrand formula applied to each hamiltonian $H_R(\Phi)$ and H_E states

$$\begin{aligned} &\langle x \pm 1, y|g'(H_R(\Phi))|x, y\rangle - \langle x \pm 1, y|g'(H_E)|x, y\rangle \\ &= \frac{1}{2\pi}\int_{\mathbf{R^2}} dudw\partial_{\bar{z}}\tilde{g'}(z)\langle x \pm 1, y|\frac{1}{z - H_R(\Phi)} - \frac{1}{z - H_E}|x, y\rangle \end{aligned} \tag{A.17}$$

We remark that the only non-vanishing matrix elements of $H_R(\Phi) - H_E$ are $t_{r,r'}$ with $r = (x', 0)$, $r' = (x', -1)$ and $|x - x'| > 2\sqrt{R}$. Thus using the resolvent formula and (A.16), (A.17) can be estimated by

$$\begin{aligned} &\frac{C}{2\pi}\int_S dudw|w|^N \sum_{|x-x'|>2\sqrt{R}} |\langle x \pm 1, y|\frac{1}{z - H_R(\Phi)}|x', 0\rangle\langle x', -1|\frac{1}{z - H_E}|x, y\rangle| \\ &\le \frac{C}{2\pi}\int_S dudw|w|^{N-2} \sum_{|x-x'|>2\sqrt{R}} e^{-C|w|(y+|x-x'|)} \\ &\le \sum_{|x-x'|>2\sqrt{R}} \frac{C}{(y + |x - x'|)^{N-3}} \le \frac{C}{(y + 2\sqrt{R})^{N-4}} \end{aligned} \tag{A.18}$$

In the first inequality we used the Combes-Thomas estimates in the form

$$|\langle x \pm 1, y|\frac{1}{z - H_R(\Phi)}|x', 0\rangle| \le \frac{C}{|w|}e^{-C|w|(y+|x-x'|)} \tag{A.19}$$

and

$$|\langle x', -1|\frac{1}{z - H_E}|x, y\rangle| \le \frac{C}{|w|}e^{-C|w|(y+|x-x'|)} \tag{A.20}$$

Proof of Lemma 2.

We remark that

$$\begin{aligned}\frac{\partial}{\partial\Phi}g(E_l(\Phi)) =& g'(E_l(\Phi))\frac{\partial E_l(\Phi)}{\partial\Phi} = g'(E_l(\Phi))\langle\Psi_l(\Phi)|\frac{\partial H_R(\Phi)}{\partial\Phi}|\Psi_l(\Phi)\rangle \\ &= \langle\Psi_l(\Phi)|g'(H_R(\Phi)\frac{\partial H_R(\Phi)}{\partial\Phi}|\Psi_l(\Phi)\rangle\end{aligned} \tag{A.21}$$

and since g' has support Δ,

$$\sigma_R(\Phi) = -\sum_{E_l(\Phi)\in\Delta}\langle\Psi_l(\Phi)|g'(H_R(\Phi)\frac{\partial H_R(\Phi)}{\partial\Phi}|\Psi_l(\Phi)\rangle = -\mathrm{Tr}g'(H_R(\Phi))\frac{\partial H_R(\Phi)}{\partial\Phi} \tag{A.22}$$

This expression is obviously gauge invariant. In (2.1) the contribution of the flux Φ is written in a "symmetric gauge". For the argument that follows it is useful to use a gauge transformation

$$\alpha_{r,r'} \to \frac{\Phi}{2}\sum_{y\geq 0}\delta_{(x-1,y),r}\delta_{(x,y),r'} + \frac{\Phi}{2}\sum_{y\geq 0}\delta_{r,(x,y)}\delta_{(x+1,y),r'} \tag{A.23}$$

In other words we concentrate the vector potential on the two rows of bonds $((x-1,y),(x,y))$, $y\geq 0$ and $((x,y),(x+1,y))$, $y\geq 0$. A particle moving around the disc $D_{x,R}$ experiences a total flux equal to Φ. We express (A.20) in terms of the matrix elements in the new gauge and find (for convenience we keep the same notation for the hamiltonians in the new gauge)

$$\sigma_R(\Phi) = \mathrm{Im}\sum_{y\geq 0}\Big(t_{r,(x-1,y)}\langle x-1,y|g'(H_R(\Phi))|x,y\rangle - t_{r,(x+1,y)}\langle x+1,y|g'(H_R(\Phi))|x,y\rangle\Big) \tag{A.24}$$

Because of Lemma A.2 we have

$$\begin{aligned}\lim_{R\to\infty}\int_0^{2\pi}\frac{d\Phi}{2\pi}\sigma_R(\Phi) = \mathrm{Im}\sum_{y\geq 0}\Big(&t_{r,(x-1,y)}\langle x-1,y|g'(H_E)|x,y\rangle \\ &- t_{r,(x+1,y)}\langle x+1,y|g'(H_E)|x,y\rangle\Big)\end{aligned} \tag{A.25}$$

Note that because of Lemmas 2, 3, 4 and 5 $\sigma_R = \mathrm{Ind}(P_F, UP_FU^\dagger)$ for every R and therefore (A.25) is independent of the function g. This enables us to replace g in (A.25) by a sequence $g_k \in C^\infty$ such that $g_k(\lambda) = 1$, $\lambda \leq \inf\Delta$, $g_k(\lambda) = 0$, $\lambda \geq \sup\Delta$, $-\frac{2}{|\Delta|} < g_k'(\lambda) < 0$, $\lambda\in\Delta$ and $\lim_{k\to\infty} g_k'(\lambda) = \frac{1}{|\Delta|}\chi_\Delta(\lambda)$.

From the spectral representation we see that $\langle r'|g_k'(H_E)|r\rangle$ tends to $\frac{1}{|\Delta|}\langle r'|P_\Delta|r\rangle$ as $k\to\infty$. Moreover by Cauchy-Schwartz

$$|\langle r'|g_k'(H_E)|r\rangle| \leq |\langle r|g_k'(H_E)^2|r\rangle|^{1/2} \leq \frac{2}{|\Delta|}\langle r|P_\Delta|r\rangle \tag{A.26}$$

Thus thanks to Lemma (A.1) and dominated convergence we see that the limit of (A.25) (with g replaced by g_k) as $k\to\infty$ is equal to the expression (1.7) of σ_E.

Appendix B.

We construct a finite rank perturbation $F_R(\Phi)$ such that the branches of $\tilde{H}_R(\Phi) = H_R(\Phi) + F_R(\Phi)$ in $\tilde{G}$ do not cross. If the levels of $H_R(0)$ are degenerate we add a potential $W_r\delta_{r,r'}$ supported on a finite number of sites and such that $\sup_r |W_r| \leq \epsilon$ with ϵ small enough, such that the levels of $H_R(0) + W$ are not degenerate. Now we examine the spectral branches associated to $H_R(\Phi) + W$. We remark that these branches necessarily cross a finite number of times. Indeed since there are a finite number of branches, the only way that there could be an infinite number of crossings would be that a pair of branches cross an infinite number of times. However the analyticity of the branches would then imply that they are in fact degenerate for all $\Phi \in [0, 2\pi]$. This possibility is excluded because the eigenvalues of $H_R(0) + W$ are not degenerate. The levels and eigenfunctions are still denoted by $E_l(\Phi)$ and $|\Psi_l(\Phi)\rangle$

Let us choose a pair of branches which cross, say $E_i(\Phi)$ and $E_j(\Phi)$. Let us label a crossing by $(i, j; \mu)$ where μ labels the the crossings belonging to the pair (i, j),

$$E_i(\Phi_{i,j;\mu}) = E_j(\Phi_{i,j;\mu}) \tag{B.1}$$

Add to the Hamiltonian the finite rank perturbation

$$\sum_{\mu} \lambda_{i,j;\mu}(\Phi)(|\Psi_i(\Phi)\rangle\langle\Psi_j(\Phi)| + |\Psi_j(\Phi)\rangle\langle\Psi_i(\Phi)|) \tag{B.2}$$

where $\lambda_{i,j;\mu}(\Phi)$ are test functions centered at $\Phi_{i,j;\mu}$ with supports that are sufficiently small so that they do not contain any other crossing. It is easily seen that in the neighborhood of $\Phi_{i,j;\mu}$ the branches of the perturbed hamiltonian satisfy

$$|\tilde{E}_i(\Phi) - \tilde{E}_j(\Phi)| = \sqrt{(E_i(\Phi) - E_j(\Phi)^2 + \lambda_{i,j;\mu}(\Phi)^2} > 0 \tag{B.3}$$

so that the perturbed hamiltonian has one less element in the set of pair of branches which cross (the amplitude of the test functions are also taken as small as needed). By iterating this procedure we end up with a hamiltonian $H_R(\Phi) + F_R(\Phi)$ with branches that do not cross. We remark that this iterative procedure allows to deal with situations where more than two branches cross at a same point by diagonalizing only 2×2 matrices.

Let N be the number of pairs of branches which cross in the initial hamiltonian $H_R(\Phi) + W$. Let δ be the smallest support of the set of all test functions. We can always choose

$$\max_{0 \leq \Phi 2\pi} |\lambda_{i,j;\mu}(\Phi)| \leq \frac{\delta^2}{N} \tag{B.4}$$

and

$$\max_{0 \leq \Phi 2\pi} |\lambda'_{i,j;\mu}(\Phi)| \leq \frac{\delta}{N} \tag{B.5}$$

Then the norm of the total perturbation $F_R(\Phi)$ satisfies

$$||F_R(\Phi)|| \leq \epsilon + \delta^2 \tag{B.6}$$

$$||\frac{\partial F_R(\Phi)}{\partial \Phi}|| \leq \delta \tag{B.7}$$

Now we can define $\tilde{\sigma}_R$ associated to $\tilde{H}_R(\Phi)$. We have $\tilde{\sigma}_R = \tilde{n}$, the number of branches of $\tilde{H}_R(\Phi)$ which cross the Fermi energy E_F. Then the same arguments leading to Lemmas 3 and 4 hold so that

$$\tilde{n} = \mathrm{Ind}\big(\tilde{U}_{T,R}(T)\tilde{P}_R(0)\tilde{U}^\dagger_{T,R}(T); U\tilde{P}_R(0)U^\dagger\big) = \mathrm{Ind}\big(U_T(T)P_F U_T^\dagger(T); UP_F U^\dagger\big) \tag{B.8}$$

Let us recall that in (B.8) the first equality is a consequence of the adiabatic theorem and the second a consequence of the stability of the relative index under compact perturbations. From Lemma 5 we have $\tilde{n} = \sigma_H$. Now from lemmas 1 and 2 we know that $n = \sigma_E$. Therefore it remains to show that in fact $\tilde{n} = n$.

Lemma B.1. Given R, we can find ϵ and δ small enough such that $\tilde{n} = n$.

Proof.

We first note that

$$\begin{aligned}
\tilde{\sigma}_R(\Phi) = \mathrm{Tr} g'(\tilde{H}_R(\Phi))\frac{\partial \tilde{H}_R(\Phi)}{\partial \Phi} &= \mathrm{Tr}\chi_\Delta(\tilde{H}_R(\Phi))g'(\tilde{H}_R(\Phi))\frac{\partial \tilde{H}_R(\Phi)}{\partial \Phi} \\
&= \mathrm{Tr}\chi_\Delta(H_R(\Phi))g'(H_R(\Phi))\frac{\partial H_R(\Phi)}{\partial \Phi} \\
&+ \mathrm{Tr}(\chi_\Delta(\tilde{H}_R(\Phi)) - \chi_\Delta(H_R(\Phi)))g'(\tilde{H}_R(\Phi))\frac{\partial \tilde{H}_R(\Phi)}{\partial \Phi} \\
&+ \mathrm{Tr}\chi_\Delta(\tilde{H}_R(\Phi))(g'(\tilde{H}_R(\Phi)) - g'(H_R(\Phi)))\frac{\partial H_R(\Phi)}{\partial \Phi} \\
&+ \mathrm{Tr}\chi_\Delta(H_R(\Phi))g'(\tilde{H}_R(\Phi))\frac{\partial \tilde{F}_R(\Phi)}{\partial \Phi}
\end{aligned} \tag{B.9}$$

In the last member of (B.9), the average over $\Phi \in [0, 2\pi]$ of the first term is $\sigma_R = n$, an integer. We show that the other three terms are smaller than 1 for small enough ϵ and δ, so that their average has to vanish since the left hand side of (B.9) has an average equal to $\tilde{n}$, also an integer.

We will use the following inequalities. First

$$||\chi_\Delta(\tilde{H}_R(\Phi)) - \chi_\Delta(H_R(\Phi))|| \leq C||F_R(\Phi)|| \leq C(\delta + \epsilon^2) \tag{B.10}$$

which follows from the functional calculus, the resolvent identity and (B.6). Second

$$||g'(\tilde{H}_R(\Phi)) - g'(H_R(\Phi))|| \leq C||F_R(\Phi)|| \leq C(\delta + \epsilon^2) \tag{B.11}$$

which follows from the Helfer-Sjostrand formula (similarly than in Appendix A). Third the trace-norms satisfy

$$||g'(\tilde{H}_R(\Phi))||_1 \leq \frac{2}{|\Delta|}\max_{0\leq\Phi\leq 2\pi}\tilde{L}_R(\Phi), \qquad ||\chi_\Delta(H_R(\Phi))||_1 \leq \max_{0\leq\Phi\leq 2\pi} L_R(\Phi) \tag{B.12}$$

where $\tilde{L}_R(\Phi)$ and $L_R(\Phi)$ are the finite number of eigenvalues of $\tilde{H}_R(\Phi)$ and $H_R(\Phi)$ in Δ.

The second, third and fourth traces on the right hand side of (B.9) are bounded above by

$$||\chi_\Delta(\tilde{H}_R(\Phi)) - \chi_\Delta(H_R(\Phi))||||g'(\tilde{H}_R(\Phi))||_1||\frac{\partial \tilde{H}_R(\Phi)}{\partial\Phi}|| \leq \frac{2C}{|\Delta|R} \max_{0\leq\Phi\leq 2\pi} \tilde{L}_R(\Phi)(\epsilon+\delta^2) \tag{B.13}$$

$$||\chi_\Delta(H_R(\Phi))||_1||g'(\tilde{H}_R(\Phi)) - g'(H_R(\Phi))||||\frac{\partial H_R(\Phi)}{\partial\Phi}|| \leq \frac{C}{R} \max_{0\leq\Phi\leq 2\pi} L_R(\Phi)(\epsilon+\delta^2) \tag{B.14}$$

$$||\chi_\Delta(H_R(\Phi))||_1||g'(\tilde{H}_R(\Phi))||||\frac{\partial F_R(\Phi)}{\partial\Phi}|| \leq \frac{2}{|\Delta|} \max_{0\leq\Phi\leq 2\pi} L_R(\Phi)\delta \tag{B.15}$$

The claim of Lemma B.1 follows by choosing δ and ϵ sufficiently small so that (B.13-15) become strictly smaller than 1.

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