Final Exam: Solutions

Quiz. (9 points)
Please answer directly on the page below. No justifications required here. A single possible answer per question. Correct answer = +1.5 points, wrong answer = -0.5 point, no answer = 0 point.

Let \((Z_n, n \geq 1)\) be i.i.d. random variables such that \(P(Z_n = +1) = P(Z_n = -1) = \frac{1}{2}\) for all \(n \geq 1\), and let us consider the following four processes:

A. \(X_0 = 0, X_{n+1} = X_n + Z_{n+1}, n \geq 0\)  
B. \(X_n = Z_n, n \geq 0\)  
C. \(X_0 = 1, X_{n+1} = X_n \cdot Z_{n+1}, n \geq 0\)  
D. \(X_0 = 0, X_1 = Z_1, X_{n+1} = Z_n + Z_{n+1}, n \geq 1\)

q1) Which of the above four processes is not a Markov chain? Answer: D

q2) Which of the above four processes does not have a finite state space? Answer: A

q3) Does there exist a constant \(c > 0\) and a distribution \(\pi\) on \(\mathbb{N}^* = \{1, 2, 3, \ldots\}\) such that \(\pi_n = \frac{c}{n}\) for every \(n \geq 1\)? Answer: □ yes ■ no

q4) Let \(i\) be a given state of a Markov chain \((X_n, n \geq 0)\) and \(T_i\) be the first return time to state \(i\). Assume moreover there exists a constant \(c > 0\) such that

\[
P(T_i = n \mid X_0 = i) = \frac{c}{n^2} \quad \text{for all } n \geq 1 \quad \text{and} \quad P(T_i = +\infty \mid X_0 = i) = 0
\]

Then state \(i\) is: □ positive-recurrent ■ null-recurrent □ transient

Let \(X\) be an irreducible and aperiodic Markov chain with state space \(\mathcal{S}\) and stationary distribution \(\pi\) such that \(\pi_i > 0\) for every \(i \in \mathcal{S}\).

q5) Which of the following two assertions is correct?

■ Without any further assumption, we know that \(X\) is positive-recurrent.

□ If \(\mathcal{S}\) is finite, then we know that \(X\) is positive-recurrent, but in the general case, we do not know.

q6) Which of the following two assertions is correct?

■ Without any further assumption, we know that \(\pi\) is also a limiting distribution.

□ If \(\pi\) satisfies the detailed balance equation, then we know that \(\pi\) is also a limiting distribution, but in the general case, we do not know.
Exercise 1. (30 points)
Let us consider the Markov chain \((X_n, n \geq 0)\) with state space \(S = \{0, 1, 2\}\) and transition graph:

where the parameters \(p, q\) both satisfy \(0 \leq p, q \leq 1\).

\(a1, 3\) points) Compute the set
\[ D_1 = \{(p, q) \in [0, 1]^2 : \text{the chain } X \text{ with parameters } p, q \text{ is irreducible}\} \]
**Answer:** In order for the chain to be irreducible, all states must communicate, which cannot happen when either \(p = 0\) or \(q = 0\), so
\[ D_1 = \{(p, q) \in [0, 1]^2 : p > 0 \text{ and } q > 0\} \]
Note that when \(q = 1\) (and \(p > 0\)), states 1 and 2 do not communicate directly, but communicate via state 0, so the chain is also irreducible in this case.

\(a2, 3\) points) Compute the set
\[ D_2 = \{(p, q) \in [0, 1]^2 : \text{the chain } X \text{ with parameters } p, q \text{ is both irreducible and aperiodic}\} \]
**Answer:** First note that \(D_2\) is necessarily a subset of \(D_1\). In order for the chain to be periodic, we need \(p = 1\) (no self-loop) and also \(q = 1\), otherwise there would always be two return paths of length 2 and 3 to any given state. We therefore obtain
\[ D_2 = \{(p, q) \in [0, 1]^2 : p > 0 \text{ and } q > 0 \text{ and } (p, q) \neq (1, 1)\} \]

\(a3, 2\) points) Explain why the chain \(X\) is ergodic when \((p, q) \in D_2\).
**Answer:** The chain is in this case irreducible and aperiodic, so also positive-recurrent (because finite), and therefore ergodic.

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From now on, let us assume that the couple of parameters \((p, q)\) belongs to the set \(D_2\).

\(b1, 5\) points) Compute the stationary distribution \(\pi\) of the chain.
**Answer:** The transition matrix \(P\) of the chain is given by
\[
P = \begin{pmatrix}
1 - p & p/2 & p/2 \\
p/2 & 1 - q & 0 \\
p/2 & q & 0
\end{pmatrix}
\]
A quick way to compute the stationary distribution $\pi$ is to observe that, by symmetry of the equation $\pi = \pi P$, $\pi_1 = \pi_2$. From that same equation, we deduce that

$$\pi_0 (1 - p) + \pi_1 2q = \pi_0 \quad \text{i.e.} \quad \pi_1 = \frac{p}{2q} \pi_0$$

which, together with the relation $\pi_0 + 2\pi_1 = 1$, gives

$$\pi_0 = \frac{q}{p + q} \quad \text{and} \quad \pi_1 = \frac{p/2}{p + q} \quad \pi_2 = \frac{p}{p + q}$$

**b2, 2 points** Is the stationary distribution $\pi$ also a limiting distribution for all $(p,q) \in D_2$?

**Answer:** Yes, as the chain is ergodic (a3).

**b3, 2 points** Is the detailed balance equation satisfied for all $(p,q) \in D_2$?

**Answer:** Yes: $\pi_0 p/2 = \pi_1 q = \pi_2 q$ and $\pi_1 (1 - q) = \pi_2 (1 - q)$. Of course, solving the detailed balance equation might have been tried in part b1) already, in which case the answer is immediate.

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**c1, 6 points** Compute the eigenvalues $(\lambda_0, \lambda_1, \lambda_2)$ of the transition matrix $P$ of the chain.

**Hint:** In order to simplify computations, it might help here to define $r = 1 - q$ and compute everything in terms of the parameters $(p,r)$ rather than $(p,q)$, and then translate back the expressions obtained in terms of the parameters $(p,q)$.

**Answer:** Following the hint and defining $r = 1 - q$, we obtain

$$\lambda_0 + \lambda_1 + \lambda_2 = \text{Trace}(P) = 1 - p \quad \text{and} \quad \lambda_0 \cdot \lambda_1 \cdot \lambda_2 = \text{det}(P) = r (p - r)$$

Remembering that $\lambda_0 = 1$ always, we obtain the following quadratic equation for the remaining two eigenvalues:

$$\lambda^2 + p\lambda + r (p - r) = 0$$

which gives $\lambda_{\pm} = -p \pm \sqrt{p^2 - 4r(p - r)} = -p \pm (p - 2r) = -r$ or $r - p$. Reverting back to $q = 1 - r$, we obtain that

$$(\lambda_0, \lambda_1, \lambda_2) = (1, q - 1, 1 - p - q)$$

Note that these eigenvalues are not necessarily ordered, depending on the values of $p$ and $q$.

**NB:** An alternate way to proceed is to note that, again by the symmetry of the matrix $P$, $u = (0, +1, -1)^T$ is an eigenvector of $P$, with corresponding eigenvalue $q - 1$. Using then the fact that Trace$(P) = 1 - p$ and $\lambda_0 = 1$, we can conclude without having to compute the determinant.

**c2, 2 points** Compute the corresponding spectral gap $\gamma$. You may leave the expression for $\gamma$ in the form $\gamma = \min(\ldots, \ldots)$ or $\gamma = \max(\ldots, \ldots)$.

**Answer:** $\gamma = \min((q-1)-(1-p), p+q, (1 - p - q) - (-1)) = \min(q, p+q, 2 - p - q) = \min(q, 2 - p - q)$, as $p + q$ is always greater than $q$. 3
d1, 3 points) Compute the set
\[ D_3 = \{(p, q) \in D_2 : \text{the stationary distribution } \pi \text{ with parameters } (p, q) \text{ is uniform on } S\} \]

**Answer:** In order to have \( \pi = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \), we need
\[
\frac{q}{p+q} = \frac{1}{3} \quad \text{i.e.} \quad 3q = p + q \quad \text{i.e.} \quad p = 2q
\]
So
\[ D_3 = \{(p, q) \in [0, 1]^2 : 0 < p \leq 1, 0 < q \leq \frac{1}{2} \text{ and } p = 2q\} \]

d2, 2 points) For what values of the parameters \((p, q) \in D_3\) is the spectral gap \(\gamma\) the largest?

**Answer:** When \(p = 2q\), we have \(\gamma = \min(q, 2 - 3q)\), which is maximum when \(q = \frac{1}{2}\) and therefore \(p = 1\) (which corresponds by the way to the symmetric circular random walk on \(S = \{0, 1, 2\}\)). In this case, the spectral gap \(\gamma = \frac{1}{2}\).

Exercise 2. (21 points)

On the state space \(S = \{0, \ldots, N\}\) with \(N \geq 2\), we would like to use the Metropolis algorithm in order to sample from the distribution \(\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{N-1}, \pi_N)\), where we assume that
\[ \pi_0 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_{N-1} \geq \pi_N > 0 \]
We start from the base chain with transition matrix \(\Psi\) and corresponding transition graph

\[\begin{array}{cccccc}
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
0 & 1 & 2 & \ldots & N-1 & N \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
\end{array}\]

a1, 3 points) Establish the list of hypotheses satisfied by this base chain that guarantee the convergence of the Metropolis algorithm.

**Answer:** The base chain is irreducible, aperiodic and for all states \(i \neq j\), \(\psi_{ij} > 0\) if and only if \(\psi_{ji} > 0\).

a2, 6 points) Compute the transition matrix \(P\) of the corresponding Metropolis chain. Do not forget to compute the values of the diagonal elements \(p_{ii}\) for \(0 \leq i \leq N\).

**Answer:** Remember that for \(j \neq i\), \(p_{ij} = \psi_{ij} a_{ij}\), where the acceptance probabilities are given by \(a_{ij} = \min(1, \frac{\pi_j}{\pi_i})\), as the base chain is symmetric here. We therefore have for \(0 \leq i \leq N-1\):
\[ a_{i,i+1} = \min(1, \frac{\pi_{i+1}}{\pi_i}) = \frac{\pi_{i+1}}{\pi_i} \quad \text{(as } \pi_{i+1} \leq \pi_i \text{ by assumption)}\]
and for $1 \leq i \leq N$:
\[
a_{i,i-1} = \min(1, \frac{\pi_{i-1}}{\pi_i}) = 1 \quad \text{(as } \pi_{i-1} \geq \pi_i \text{ by assumption)}
\]
Correspondingly, we obtain
\[
p_{i,i+1} = \frac{\pi_{i+1}}{2\pi_i} \quad \text{for } 0 \leq i \leq N - 1
\]
\[
p_{i,i-1} = \frac{1}{2} \quad \text{for } 1 \leq i \leq N
\]
\[
p_{i,i} = \frac{\pi_i - \pi_{i+1}}{2\pi_i} \quad \text{for } 1 \leq i \leq N - 1
\]
\[
p_0,0 = \frac{2\pi_0 - \pi_1}{\pi_0} \quad \text{and} \quad p_{N,N} = \frac{1}{2}
\]
Let us further assume that for $0 \leq i \leq N$,
\[
\pi_i = \frac{c^i}{Z} \quad \text{where } 0 < c < 1 \quad \text{and} \quad Z = \sum_{i=0}^{N} c^i = \frac{1 - c^{N+1}}{1 - c} \quad \text{(1)}
\]
a3, 2 points) Compute the transition matrix $P$ in this particular case.

**Answer:** Following what was done above, we obtain (note that the value of $Z$ is of no use here, as only the ratio $\frac{\pi_{i+1}}{\pi_i} = c$ enters into the formulas):
\[
p_{i,i+1} = \frac{c}{2} \quad \text{for } 0 \leq i \leq N - 1
\]
\[
p_{i,i-1} = \frac{1}{2} \quad \text{for } 1 \leq i \leq N
\]
\[
p_{i,i} = \frac{1 - c}{2} \quad \text{for } 1 \leq i \leq N - 1
\]
\[
p_{0,0} = \frac{2 - c}{2} \quad \text{and} \quad p_{N,N} = \frac{1}{2}
\]
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Still under the assumption that the distribution $\pi$ is given by formula (1), imagine now that we start the Metropolis algorithm from the following modified base chain with transition matrix $\tilde{\Psi}$ and corresponding transition graph

b1, 2 points) Are the hypotheses listed in question a1) still satisfied here?

**Answer:** Yes: even though the matrix $\Psi$ is not anymore symmetric, it still holds that for all $i \neq j$, $\psi_{ij} > 0$ if and only if $\psi_{ji} > 0$ (and the chain remains irreducible and aperiodic).
b2, 6 points) Compute the transition matrix $\tilde{P}$ of the corresponding Metropolis chain.

Answer: As $\Psi$ is not anymore symmetric, the acceptance probabilities are now given by the formula

$$\tilde{a}_{ij} = \min \left( 1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}} \right)$$

which leads to, for $0 \leq i \leq N - 1$:

$$\tilde{a}_{i,i+1} = \min \left( 1, \frac{\pi_{i+1} \psi_{i+1,i}}{\pi_i \psi_{i,i+1}} \right) = \min(1, 2c)$$

and for $1 \leq i \leq N$:

$$\tilde{a}_{i,i-1} = \min \left( 1, \frac{\pi_{i-1} \psi_{i-1,i}}{\pi_i \psi_{i,i-1}} \right) = \min \left( 1, \frac{1}{2c} \right)$$

From there, we deduce that

$$\tilde{p}_{i,i+1} = \frac{1}{3} \min(1, 2c) \quad \text{for } 0 \leq i \leq N - 1$$

$$\tilde{p}_{i,i-1} = \frac{2}{3} \min \left( 1, \frac{1}{2c} \right) \quad \text{for } 1 \leq i \leq N$$

So for $c \leq \frac{1}{2}$, we have

$$\tilde{p}_{i,i} = \frac{1 - 2c}{3} \quad \text{for } 1 \leq i \leq N - 1 \quad \tilde{p}_{0,0} = \frac{3 - 2c}{3} \quad \text{and} \quad \tilde{p}_{N,N} = \frac{1}{3}$$

while for $c \geq \frac{1}{2}$, we have

$$\tilde{p}_{i,i} = \frac{2 - \frac{1}{c}}{3} \quad \text{for } 1 \leq i \leq N - 1 \quad \tilde{p}_{0,0} = \frac{2}{3} \quad \text{and} \quad \tilde{p}_{N,N} = \frac{3 - \frac{1}{c}}{3}$$

b3, 2 points) For what value of $0 < c < 1$ does the Metropolis algorithm introduces the least number of self-loops in the chain?

Answer: From the above formulas, we see that this happens when $c = \frac{1}{2}$ (this is actually the case where the distribution $\pi$ is the stationary distribution of the base chain with transition matrix $\Psi$, so that the Metropolis algorithm always accepts the proposed moves).