Solutions 5

1. a) This chain is clearly ergodic. The transition matrix is

\[
\begin{pmatrix}
1 - p & p & 0 \\
1/2 & 0 & 1/2 \\
0 & p & 1 - p
\end{pmatrix}
\]

Assume that the detailed balance equation is satisfied. Then

\[
\pi_1^*/2 = \pi_2^* p = \pi_0^* p
\]

We conclude that

\[
\pi_0^* = \pi_2^* = \frac{1}{2(1+p)} \quad \pi_1^* = \frac{p}{1+p}
\]

It is then easy to verify that \( \pi^* = \pi^* P \), and so this is indeed a stationary distribution, which obviously satisfies the detailed balance equation.

b) We know that \( \lambda_0 = 1 \), and so, to compute the eigenvalues, we must solve the equations

\[
\begin{align*}
2 - 2p &= 1 + \lambda_1 + \lambda_2 \\
-p(1 - p) &= \lambda_1 \lambda_2
\end{align*}
\]

Solving this, we obtain that \( \lambda_1 = 1 - p \) and \( \lambda_2 = -p \). So \( \lambda_* = \max(p, 1 - p) \) and the spectral gap is given by \( \gamma = 1 - \lambda_* = \min(p, 1 - p) \).

c) For \( p = \frac{1}{N} \), the spectral gap is \( \gamma = \frac{1}{N} \). From the theorem seen in class, we know that \( \|P^n_i - \pi\|_{TV} \leq \frac{\exp(-\gamma n)}{2\sqrt{\pi_i}} \), so here,

\[
\max_{i \in S} \|P^n_i - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\frac{1 + 1/N}{1/N}} \exp(-n/N) \leq \sqrt{N} \exp(-n/N) = \exp\left(\frac{\log N}{2} - \frac{n}{N}\right)
\]

Taking therefore \( n \geq N \left(\frac{\log N}{2} + c\right) \) with \( c > 0 \) sufficiently large (more precisely, \( c = \log(1/\varepsilon) \)) ensures that the maximum total variation norm is below \( \varepsilon \).

d) For \( p = 1 - \frac{1}{N} \), the spectral gap is again \( \gamma = \frac{1}{N} \). So

\[
\max_{i \in S} \|P^n_i - \pi\|_{TV} \leq \frac{1}{2} \sqrt{2(2 - 1/N)} \exp(-n/N) \leq \exp(-n/N)
\]

Taking therefore \( n \geq cN \) with \( c = \log(1/\varepsilon) \) ensures that the maximum total variation norm is below \( \varepsilon \), so

\[
T_\varepsilon \leq N \log(1/\varepsilon)
\]
2. a) The transition matrix being doubly stochastic, the stationary distribution is uniform (i.e. \( \pi_i = \frac{1}{2N} \) for every \( i \in S \)) and satisfies the detailed balance equation.

b) Solving the equation \( P\phi^{(1)} = \lambda\phi^{(1)} \), we obtain

\[
\begin{align*}
\frac{N-1}{N} a + \frac{1}{N} b &= \lambda a \\
\frac{N-1}{N} a - \frac{1}{N} b &= \lambda b
\end{align*}
\]

which is saying that \( \lambda \) is an eigenvalue of the \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
\frac{N-1}{N} & \frac{1}{N} \\
\frac{N-1}{N} & -\frac{1}{N}
\end{pmatrix}
= \begin{pmatrix}
1 - \delta & \delta \\
1 - \delta & -\delta
\end{pmatrix}
\]

where we have set \( \delta = \frac{1}{N} \). These eigenvalues are given by

\[
\lambda_{\pm} = \frac{1 - 2\delta \pm \sqrt{(1 - 2\delta)^2 + 8\delta(1 - \delta)}}{2} = \frac{1 - 2\delta \pm \sqrt{1 + 4\delta - 4\delta^2}}{2}
\]

For \( \delta \) small (i.e. \( N \) large), the largest of these 2 eigenvalues is \( \lambda_+ \), which is approximately given by

\[
\lambda_+ \approx \frac{1 - 2\delta + (1 + 2\delta - 4\delta^2)}{2} = 1 - 2\delta^2 = 1 - \frac{2}{N^2}
\]

so the spectral gap \( \gamma \approx \frac{2}{N^2} \).

c) By the theorem seen in class,

\[
\max_{i \in S} \| P^n_i - \pi \|_{TV} \leq \frac{\sqrt{2N}}{2} \exp(-\gamma n) \leq \sqrt{2} \exp\left(\frac{\log N}{2} - \frac{2n}{N^2}\right)
\]

is below \( \varepsilon \) for \( n \geq \frac{N^2}{2} \left( \frac{\log N}{2} + \log(\sqrt{2}/\varepsilon) \right) \), so

\[
T_\varepsilon \leq \frac{N^2}{2} \left( \frac{\log N}{2} + \log(\sqrt{2}/\varepsilon) \right)
\]