Solutions of the last three exercises of Homework 1

Exercise 3.  a) The transition matrix is given by

\[
P = \begin{pmatrix}
0 & p & 0 & 1-p \\
1-q & 0 & q & 0 \\
p & 0 & 1-p & 0 \\
1-q & 0 & q & 0
\end{pmatrix}.
\]

Case 1.  \(p = q = 1\)

b) There are three equivalence classes: \(\{1\}, \{4\} \) and \(\{2,3\}\).

c) The class \(\{2,3\}\) is periodic of period 2.

d) The classes \(\{1\}\) and \(\{4\}\) are transient, the class \(\{2,3\}\) is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

\[
\pi = (0, 1/2, 1/2, 0).
\]

Case 2.  \(p = 1, q = 0\)

b) There are three equivalence classes: \(\{3\}, \{4\}\) and \(\{1,2\}\).

c) The class \(\{1,2\}\) is periodic of period 2.

d) The classes \(\{3\}\) and \(\{4\}\) are transient, the class \(\{1,2\}\) is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

\[
\pi = (1/2, 1/2, 0, 0).
\]

Case 3.  \(0 < p, q < 1\)

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) Because the chain is irreducible, the stationary distribution is unique and given by

\[
\pi = \left(\frac{1-q}{2}, p, q, \frac{1-p}{2}\right).
\]

(note that this expression matches the former two cases).

f) Because the chain is periodic, the stationary distribution is not a limiting distribution.

g) The detailed balance equations are satisfied for all values of \(0 < p, q < 1\).
Exercise 4. a) The transition matrix is given by

\[
P = \begin{pmatrix}
0 & p & 0 & 1-p \\
1-q & 0 & q & 0 \\
0 & p & 0 & 1-p \\
q & 0 & 1-q & 0
\end{pmatrix}.
\]

Case 1. \( p = q = 1 \)

b) There are three equivalence classes: \( \{1\} \), \( \{4\} \) and \( \{2,3\} \) (but note the graph is different from ex. 3, same case).

c) The class \( \{2,3\} \) is periodic of period 2.

d) The classes \( \{1\} \) and \( \{4\} \) are transient, the class \( \{2,3\} \) is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

\[
\pi = (0, 1/2, 1/2, 0).
\]

Case 2. \( p = 1, q = 0 \)

b) There are three equivalence classes: \( \{3\} \), \( \{4\} \) and \( \{1,2\} \) (but note the graph is different from ex. 3, same case).

c) The class \( \{1,2\} \) is periodic of period 2.

d) The classes \( \{3\} \) and \( \{4\} \) are transient, the class \( \{1,2\} \) is recurrent.

e) The stationary distribution is unique (the chain is not irreducible, but there is a single recurrent class) and given by

\[
\pi = (1/2, 1/2, 0, 0).
\]

Case 3. \( 0 < p, q < 1 \)

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) Because the chain is irreducible, the stationary distribution is unique and given by

\[
\pi = \left( \frac{p + q - 2pq}{2}, \frac{p}{2}, \frac{1 - p - q + 2pq}{2}, \frac{1 - p}{2} \right).
\]

(note that this expression matches the former two cases).

f) Because the chain is periodic, the stationary distribution is not a limiting distribution.

g) The detailed balance equations are satisfied for all values of \( 0 < p < 1 \), but \( q = \frac{1}{2} \) only.

Exercise 5. a) The transition matrix is given by

\[
P = \begin{pmatrix}
0 & 1-p & p & 0 \\
q & 0 & 0 & 1-q \\
1-q & 0 & 0 & q \\
0 & p & 1-p & 0
\end{pmatrix}.
\]
Case 1. $p = q = 1$

b) The chain is irreducible.

c) The chain is periodic of period 4.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) The matrix $P$ is doubly stochastic and the chain is irreducible. Hence, the stationary distribution is unique and it is the uniform distribution, i.e.,

$$\pi = (1/4, 1/4, 1/4, 1/4).$$

Case 2. $p = 1, q = 0$

b) There are two equivalence classes: $\{1, 3\}$ and $\{2, 4\}$.

c) Both equivalence classes are periodic of period 2.

d) Both equivalence classes are recurrent.

e) The matrix $P$ is doubly stochastic, but the chain is not irreducible, so there are multiple stationary distributions, given by

$$\pi = (\alpha/2, \beta/2, \alpha/2, \beta/2).$$

with $0 \leq \alpha, \beta \leq 1$, $\alpha + \beta = 1$.

Case 3. $0 < p, q < 1$

b) The chain is irreducible.

c) The chain is periodic of period 2.

d) Because the chain is finite and irreducible, it is (positive-)recurrent.

e) The matrix $P$ is doubly stochastic and the chain is irreducible. Hence, the stationary distribution is unique and it is the uniform distribution, i.e.,

$$\pi = (1/4, 1/4, 1/4, 1/4).$$

f) Because the chain is periodic, the stationary distribution is not a limiting distribution.

g) Since the stationary distribution is the uniform distribution, the detailed balance equations are satisfied if and only if $p + q = 1$.

**Answer to the final question.** Of course, the matrix $P$ itself and the equivalence classes do depend on the labelling of the states, as well as the expression of the stationary distribution(s) $\pi$. But the questions related to periodicity, recurrence, existence and uniqueness of the stationary distribution, limiting distribution and detailed balance are independent of the labelling of the states.