1. For this exercise, let \((U_n, n \geq 1)\) be a sequence of i.i.d. \(\sim U([0, 1])\) random variables.

**First case.** \(X_0 = 0, Y_0 = 1\).

a) One coupling that maximizes the chances of \(X\) and \(Y\) to meet after the first step is described as follows:

\[
\begin{aligned}
\text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \quad \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n \\
\text{if } \frac{1}{4} < U_{n+1} \leq \frac{1}{2} & \quad \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n - 1 \\
\text{if } \frac{1}{2} < U_{n+1} \leq \frac{3}{4} & \quad \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\
\text{if } \frac{3}{4} < U_{n+1} \leq 1 & \quad \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1
\end{aligned}
\]

With this coupling, the probability that \(X\) and \(Y\) meet after one step is \(\frac{1}{2}\), which can be seen to be the maximum.

b) Let \(\xi_{n+1}\) be the random variable defined as

\[
\xi_{n+1} = \begin{cases} 
+1 & \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} \\
0 & \text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} \\
-1 & \text{if } \frac{3}{4} < U_{n+1} \leq 1
\end{cases}
\]

If both \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n + \xi_{n+1}\), then the two chains never meet.

But another option is also to have \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n - \xi_{n+1}\).

**Variant:** \(X_0 = 0, Y_0 = 2\).

a) In this case, one coupling that maximizes the chances of \(X\) and \(Y\) to meet after the first step is:

\[
\begin{aligned}
\text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \quad \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n - 1 \\
\text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} & \quad \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\
\text{if } \frac{3}{4} < U_{n+1} \leq 1 & \quad \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1
\end{aligned}
\]

With this coupling, the probability that \(X\) and \(Y\) meet after one step is \(\frac{1}{4}\), which can be seen to be the maximum (NB: This coupling can also be described with the random variable \(\xi_{n+1}\) above: \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n - \xi_{n+1}\)).

b) In this case, only the coupling \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n + \xi_{n+1}\) ensures that the walks never meet. There is no other coupling guaranteeing this property.