Solutions 1

1. Using Stirling’s approximation for \( \binom{2n}{n} = \frac{2^n}{n!} \), we obtain

\[
\binom{2n}{n} p^n q^n \sim \frac{\sqrt{2\pi(2n)}}{2\pi n \left( \frac{n}{2} \right)^{2n}} (pq)^n = \left( \frac{4pq}{\sqrt{\pi n}} \right)^n
\]

2. a) Both \( X \) and \( Y \) are random walks with probability 1/4 to go in either direction, and probability 1/2 to stay in place.

b) No, they are not independent: when \( X \) makes a move, \( Y \) does not, and vice-versa.

c) Both \( U \) and \( V \) are simple symmetric random walks with probability 1/2 to go in either direction.

d) Yes, they are independent. Denote \( U_n = \eta_1 + \ldots + \eta_n, V_n = \chi_1 + \ldots + \chi_n \). Then one can check e.g. that (and similarly for all \( \pm 1 \) combinations)

\[
P(\eta_n = +1, \chi_n = +1) = P(\xi_n = (+1,0)) = \frac{1}{4} = P(\eta_n = +1) \cdot P(\chi_n = +1)
\]

e) Note that \( \overrightarrow{S}_{2n} = (0,0) \) if and only if \( U_{2n} = V_{2n} = 0 \), so by the independence shown above, we obtain

\[
P(\overrightarrow{S}_{2n} = (0,0) \mid \overrightarrow{S}_0 = (0,0)) = P(U_{2n} = 0, V_{2n} = 0 \mid U_0 = 0, V_0 = 0)
\]

\[
= P(U_{2n} = 0 \mid U_0 = 0) \cdot P(V_{2n} = 0 \mid V_0 = 0) = \left( \binom{2n}{n} 2^{-2n} \right)^2 \sim \frac{1}{\pi n}
\]

by Exercise 1.

The solutions of exercises 3-5 will be given in 4 weeks from now.