Exercise 1. (11 points) Let \( f : \mathbb{N} \rightarrow \mathbb{R} \) be a function such that \( f(0) = 1 \) and \( 0 < f(i) < 1 \) for \( i \geq 1 \). Consider the Markov chain with state space \( S = \mathbb{N} \) and transition probabilities

\[
p_{ij} = \begin{cases} 
  f(i), & j = i + 1, \quad i \geq 0 \\
  1 - f(i), & j = i - 1, \quad i \geq 1.
\end{cases}
\]

a) Is the chain irreducible? Justify your answer using the Chapman-Kolmogorov equation.

b) Consider two initial probability distributions \( \pi_{j}^{(0)} = \mu_{j} \) and \( \pi_{j}^{(0)} = \nu_{j} \) where \( \mu_{2k} = 0, \nu_{2k} \neq 0 \) and \( \mu_{2k+1} \neq 0, \nu_{2k+1} = 0, \ k \in \mathbb{N} \). We set \( \mu^{(n)} = \mu P^n, \nu^{(n)} = \nu P^n \). Prove that for any time \( n \geq 1 \)

\[
\|\mu^{(n)} - \nu^{(n)}\|_{TV} = 1
\]

c) We recall the “ratio test” for the convergence of a series \( \sum_{j \in \mathbb{N}} a_{j} \) of positive terms \( a_{j} > 0 \). Let \( \lim_{j \to +\infty} \frac{a_{j+1}}{a_{j}} = L \). If \( L < 1 \) the series converges; if \( L > 1 \) the series diverges; and if \( L = 1 \) the test is inconclusive. Prove that the stationary distribution exists if

\[
\lim_{j \to +\infty} \frac{f(j)}{1 - f(j + 1)} < 1 
\]

and does not exist if

\[
\lim_{j \to +\infty} \frac{f(j)}{1 - f(j + 1)} > 1
\]

Does the stationary distribution exist for the function \( f(0) = 1, f(i) = \frac{1}{2}, i \geq 1 \) ?

d) In each case (1) and (2) in the above question: Is the chain positive recurrent? Is the chain ergodic?

e) Let now \( 0 < p, q < 1 \) and consider the case where \( f(0) = 1, f(2k) = p \) and \( f(2k - 1) = q \) for every \( k \geq 1 \). For what values of \( p \) and \( q \) does the chain admit a stationary distribution?
Exercise 2. (9 points) Let $N$ be an odd number greater than or equal to 3 and let $S = \{0, \ldots, N - 1\}$. Let then $(X_n, n \geq 0)$ be a Markov chain with state space $S$ and $N \times N$ transition matrix

$$P = \begin{pmatrix} 0 & a & 0 & \ldots & 0 & b \\ b & 0 & a & 0 \\ 0 & b & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & a & 0 \\ a & 0 & \ldots & 0 & b & 0 \end{pmatrix}$$

with $a, b > 0$ and $a + b = 1$.

a) Explain why the chain is ergodic for all such values of $a$ and $b$.

b) Compute its unique stationary and limiting distribution $\pi$.

c) For what values of $a$ and $b$ does detailed balance hold?

d) Among the following three matrices, which are transition matrices?

$$P^T, \quad P^T P, \quad PP^T$$

(where $P^T$ stands for the transpose of $P$).

Let now $Q = P^T P$. Even though the chain is not necessarily reversible for all values of $a$ and $b$, it can be shown here that for all $i \in S$ and $n \geq 1$,

$$\|P_i^n - \pi\|_{TV} \leq \frac{1}{2\sqrt{\pi_i}} (\lambda_\ast(Q))^{n/2}$$

(3)

where $\lambda_0(Q) \geq \lambda_1(Q) \geq \ldots \geq \lambda_{N-1}(Q)$ are the (real) eigenvalues of the $N \times N$ matrix $Q$, and

$$\lambda_\ast(Q) = \max_{k \in \{1, \ldots, N-1\}} |\lambda_k(Q)|$$

e) In the case $N = 3$, compute the eigenvalues of $Q$ and deduce the value of $\lambda_\ast(Q)$.

f) Again in the case $N = 3$, deduce from (3) an upper bound on the mixing time

$$T_\varepsilon = \inf \{n \geq 1 : \max_{i \in S} \|P_i^n - \pi\|_{TV} \leq \varepsilon \}$$

where $\varepsilon > 0$. 