Exercise 1. [Barker’s algorithm]
Let \( \pi = (\pi_i, i \in S) \) be a distribution on a finite state space \( S \) such that \( \pi_i > 0 \) for all \( i \in S \) and let us consider the base chain with transition probabilities \( \psi_{ij} \), which is assumed to be irreducible, aperiodic and such that \( \psi_{ij} > 0 \) if and only if \( \psi_{ji} > 0 \). Define the following acceptance probabilities:

\[
a_{ij} = \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij} + \pi_j \psi_{ji}}
\]
as well as a new chain with transition probabilities \( p_{ij} = \psi_{ij} a_{ij} \) if \( j \neq i \). Show that this new chain is ergodic and that it satisfies the detailed balance equation:

\[
\pi_i p_{ij} = \pi_j p_{ji}, \quad \forall i, j \in S
\]

Exercise 2. [Metropolized independent sampling in a particular case]
Let \( 0 < \theta < 1 \) and let us consider the following distribution \( \pi \) on \( S = \{1, \ldots, N\} \):

\[
\pi_i = \frac{1}{Z} \theta^{i-1}, \quad i = 1, \ldots, N
\]
where \( Z \) is the normalization constant, whose computation is left to the reader.

a) Consider the base chain \( \psi_{ij} = \frac{1}{N} \) for all \( i, j \in S \) and derive the transition probabilities \( p_{ij} \) obtained with the Metropolis-Hastings algorithm.

b) Using the result of the course, derive an upper bound on \( \|P^n_i - \pi\|_{TV} \). Compare the bounds obtained for \( i = 1 \) and \( i = N \) (for large values of \( N \)).

c) Deduce an upper bound on the (order of magnitude of the) mixing time

\[
T_\varepsilon = \inf \{n \geq 1 : \max_{i \in S} \|P^n_i - \pi\|_{TV} \leq \varepsilon \}
\]

Exercise 3. [Coupling]
The first goal of this exercise is to show that for any two distributions \( \mu \) and \( \nu \) on a common state space \( S \), there always exist two (coupled) random variables \( X \) and \( Y \) with values in \( S \) such that

\[
\mathbb{P}(X = i) = \mu_i, \quad \forall i \in S \quad \mathbb{P}(Y = j) = \nu_j, \quad \forall j \in S \quad \text{and} \quad \|\mu - \nu\|_{TV} = \mathbb{P}(X \neq Y)
\]
Remember that in general, if \( X \) and \( Y \) satisfy the first two conditions, then we only have an inequality in the third statement. We need therefore to find a proper joint distribution for \( X \) and \( Y \) such that equality holds.

a) Define first \( \xi_i = \min(\mu_i, \nu_i) \) for \( i \in S \). Note that \( \xi \) itself is not a distribution, as \( \sum_{i \in S} \xi_i \leq 1 \) in general. Show that setting \( \mathbb{P}(X = Y = i) = \xi_i \) for all \( i \in S \) implies indeed that

\[
\|\mu - \nu\|_{TV} = \mathbb{P}(X \neq Y)
\]
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b) We need now to define $\mathbb{P}(X = i, Y = j)$ for $i \neq j$ so that $\mathbb{P}(X = i) = \mu_i$, $\forall i \in S$ and $\mathbb{P}(Y = j) = \nu_j$, $\forall j \in S$. Show that the following proposal works (it is not the unique one):

$$
\mathbb{P}(X = i, Y = j) = \frac{(\mu_i - \xi_i)(\nu_j - \xi_j)}{1 - \sum_{k \in S} \xi_k}
$$

[In particular, observe that there are lots of zeros in this joint distribution: if $\mu_i \leq \nu_i$ for a given $i \in S$, then $\mathbb{P}(X = i, Y = j) = 0$ for all $j \in S \setminus i$; likewise, if $\nu_j \leq \mu_j$ for a given $j \in S$, then $\mathbb{P}(X = i, Y = j) = 0$ for all $i \in S \setminus j$. $X$ and $Y$ are therefore tightly coupled!]

NB: And what if $\sum_{k \in S} \xi_k = 1$?

c) Use this to show that for an ergodic Markov chain with transition matrix $P$ and stationary distribution $\pi$, the total variation distance $d(n) = \max_{i \in S} \| P^n_i - \pi \|_{TV}$ is a non-increasing function of $n$.

*Hint:* A new coupling is required here.