Exercise 1. [Gibbs sampling]
Let $S = \{1, \ldots, N\}$ and $d \geq 1$. We would like to sample from a distribution $\pi$ on $S^d$ defined as
\[ \pi(x) = \frac{g(x)}{Z}, \quad x \in S^d, \]
where $g$ is some positive function on $S^d$ and $Z = \sum_{x \in S^d} g(x)$ is the normalization constant, which we would like to avoid computing.

A possible way to handle this problem is the following.
1. Start from a state $x \in S^d$;
2. Choose an index $u \in \{1, \ldots, d\}$ uniformly at random;
3. Update the value of $x_u$ to $x'_u$, which is sampled from the following conditional distribution:
\[ \pi(x'_u | x_1, \ldots, x_{u-1}, x_{u+1}, \ldots, x_d) = \frac{\pi(x_1, \ldots, x_{u-1}, x'_u, x_{u+1}, \ldots, x_d)}{\sum_{y_u \in S} \pi(x_1, \ldots, x_{u-1}, y_u, x_{u+1}, \ldots, x_d)} \]
4. Repeat from 2.

What is the advantage of such a method? The above conditional probability can actually be rewritten as
\[ \pi(x'_u | x_1, \ldots, x_{u-1}, x_{u+1}, \ldots, x_d) = \frac{g(x_1, \ldots, x_{u-1}, x'_u, x_{u+1}, \ldots, x_d)}{\sum_{y_u \in S} g(x_1, \ldots, x_{u-1}, y_u, x_{u+1}, \ldots, x_d)} \]
which only requires to compute one sum and not a multidimensional one, as required for computing the normalization constant $Z$.

Your task now is to formalize slightly the above algorithm by expressing it as a Markov chain $(X_n, n \geq 0)$ on $S^d$ and
a) writing down its transition probabilities $p(x, y)$, $x, y \in S^d$;
b) showing that the detailed balance equation is satisfied, i.e. that $\pi(x) p(x, y) = \pi(y) p(y, x)$, for all $x, y \in S^d$.

Can therefore this algorithm be viewed as a Metropolis-Hastings algorithm?
Exercise 2. On the state space $S = \{0, 1, 2\}$ and given $\beta > 0$, consider the following distribution:

$$
\pi = \frac{1}{Z} \left( 1, e^{-2\beta}, e^{-\beta} \right)
$$

where the normalization constant $Z = 1 + e^{-2\beta} + e^{-\beta}$ is easy to compute in this case. For any given $\beta > 0$, we would like to sample from $\pi$, in order to obtain (by taking $\beta$ large) an estimate of the global minimum of the function $f : S \to \mathbb{Z}$ defined as $f(0) = 0$, $f(1) = 2$ and $f(2) = 1$. Of course, in this situation, both finding the global minimum of $f$ and sampling from the distribution $\pi$ are trivial tasks, but the idea here is to get an idea of the performance (i.e. rate of convergence) of the Metropolis-Hastings algorithm in a simple case.

Consider the base chain on $S$ with transition probabilities

$$
\psi_{01} = \psi_{21} = 1 \quad \text{and} \quad \psi_{10} = \psi_{12} = \frac{1}{2}.
$$

a) Compute the transition probabilities $p_{ij}$ of the corresponding Metropolis chain.

b) Check that the detailed balance equation is satisfied.

c) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2$ of $P$. (Hint: You already know that $\lambda_0 = 1$.)

d) Express the spectral gap $\gamma$ as a function of $\beta$. How does it behave as $\beta$ gets large?