Exercise 1. (22+2 points)

Useful reminders for this exercise: For any $0 < x < 1$ and $k \geq 1$, we have:

$$
\sum_{n\geq0} x^n = \frac{1}{1-x}  \\
\sum_{n\geq1} nx^{n-1} = \frac{\partial}{\partial x} \left( \sum_{n\geq1} x^n \right) = \ldots  \\
\sum_{j=1}^k x^j = \frac{x^{k+1} - x}{x - 1}
$$

Let us consider the Markov chain $(X_n, n \geq 0)$ with state space $S = \{iA, iB, i \in \mathbb{N}\}$ and the following transition graph:

![Transition Graph](image)

where $0 < p < 1$ is a fixed parameter.

a) For every $n \geq 1$, compute the value of

$$
\mathbb{P}^{(n)}_{0A,0A} = \mathbb{P}(X_n = 0A, X_{n-1} \neq 0A, \ldots, X_1 \neq 0A | X_0 = 0A)
$$

b) For what values of $0 < p < 1$ is state $0A$ recurrent? Justify your answer.

Let now $T_{0A} = \inf\{n \geq 1 : X_n = 0A\}$ be the first return time to state $0A$.

c) Compute $\mathbb{E}(T_{0A} | X_0 = 0A)$.

d) For what values of $0 < p < 1$ is state $0A$ positive-recurrent? Justify your answer.
e) Without doing any computation, explain why does the chain \((X_n, n \geq 0)\) admit a unique stationary distribution \(\pi\) for every value of \(0 < p < 1\).

f) Show by induction on \(i\) that \(\pi_{iA} = \pi_{iB}\) for every \(i \in \mathbb{N}\).

g) Use f) to compute the stationary distribution \(\pi\).

h) Are the detailed balance equations satisfied? Justify your answer.

**BONUS** For every \(n \geq 1\), compute the value of
\[
P^{(n)}_{0A,0A} = \mathbb{P}(X_n = 0A \mid X_0 = 0A)
\]

**Exercise 2.** (20+2 points)

Let \(0 < p \leq \frac{1}{2}\) and \(0 < q \leq 1\) be two fixed parameters and consider the Markov chain \((X_n, n \geq 0)\) with state space \(S = \{0, 1, 2\}\) and transition matrix
\[
P = \begin{pmatrix}
1 - 2p & p & p \\
p & 1 - q & 0 \\
p & 0 & 1 - q
\end{pmatrix}
\]

a) For any given values of \(p, q\), compute the stationary distribution \(\pi\) of the chain \(X\).

b) For any given values of \(p, q\), compute the eigenvalues of \(P\).

c) Deduce the corresponding spectral gap \(\gamma\) of the chain \(X\), as well as a tight upper bound on
\[
\|P^n_0 - \pi\|_{TV}
\]

for large values of \(n\).

Let us now consider another Markov chain \((Y_n, n \geq 0)\) with same state space \(S\) and transition matrix
\[
Q = \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{pmatrix}
\]

d) For what values of \(p, q\) do the chains \(X\) and \(Y\) share the same stationary distribution?

e) Among the values of \(p, q\) found in part d), which correspond to the largest spectral \(\gamma\) for the chain \(X\)?

**BONUS** Do the spectral gaps of \(X\) and \(Y\) match in this last case?
Exercise 3. (18 points)
Let us consider the Markov chain with state space $S = \mathbb{N}^* = \{1, 2, 3, \ldots\}$, with transition graph

and with corresponding transition matrix $\Psi$.

a) Let $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$ be a distribution on $S$ such that $\pi_i > \pi_{i+1}$ for all $i \geq 1$. Starting from the base chain with transition matrix $\Psi$, design a new Markov chain with transition matrix $P$ whose stationary distribution is $\pi$. Compute the matrix $P$ explicitly.

b) What do we know about the chain with transition matrix $P$ and the stationary distribution $\pi$? List all the properties you can think of.

c) Compute $\lim_{i \to \infty} p_{i,i+1}$ in the 3 following cases:
   
   c1) $\pi_i = \frac{1}{Z} \frac{1}{i^q}$, $i \geq 1$. Here, $q > 1$ is a fixed parameter and $Z = \sum_{i \geq 1} \frac{1}{i^q}$.
   
   c2) $\pi_i = \frac{1}{Z} \exp(-i)$, $i \geq 1$, with $Z = \sum_{i \geq 1} \exp(-i)$.
   
   c3) $\pi_i = \frac{1}{Z} \exp(-i^2)$, $i \geq 1$, with $Z = \sum_{i \geq 1} \exp(-i^2)$.

d) For which of the above 3 example(s) does the Metropolis algorithm always accept a move from $i$ to $i - 1$, $\forall i \geq 2$?