Final Exam

Surname: . . . . . . . . . . . . . . . . . . . . . . . . . . . First name: . . . . . . . . . . . . . . . . . . . . . . . . . . . Section: . . . . . . . . . . . .

Quiz. (9 points)
Please answer directly on the page below. No justifications required here. A single possible answer per question. Correct answer = +1.5 points, wrong answer = -0.5 point, no answer = 0 point.

Let \((Z_n, n \geq 1)\) be i.i.d. random variables such that \(P(Z_n = +1) = P(Z_n = -1) = \frac{1}{2}\) for all \(n \geq 1\), and let us consider the following four processes:

A. \(X_0 = 0, X_{n+1} = X_n + Z_{n+1}, n \geq 0\)
B. \(X_n = Z_n, n \geq 0\)
C. \(X_0 = 1, X_{n+1} = X_n \cdot Z_{n+1}, n \geq 0\)
D. \(X_0 = 0, X_1 = Z_1, X_{n+1} = Z_n + Z_{n+1}, n \geq 1\)

q1) Which of the above four processes is not a Markov chain? Answer: 

q2) Which of the above four processes does not have a finite state space? Answer: 

q3) Does there exist a constant \(c > 0\) and a distribution \(\pi\) on \(\mathbb{N}^* = \{1, 2, 3, \ldots\}\) such that \(\pi_n = \frac{c}{n}\) for every \(n \geq 1\)? Answer: □ yes □ no

q4) Let \(i\) be a given state of a Markov chain \((X_n, n \geq 0)\) and \(T_i\) be the first return time to state \(i\). Assume moreover there exists a constant \(c > 0\) such that

\[
P(T_i = n \mid X_0 = i) = \frac{c}{n^2} \quad \text{for all } n \geq 1 \quad \text{and} \quad P(T_i = +\infty \mid X_0 = i) = 0
\]

Then state \(i\) is: □ positive-recurrent □ null-recurrent □ transient

Let \(X\) be an irreducible and aperiodic Markov chain with state space \(S\) and stationary distribution \(\pi\) such that \(\pi_i > 0\) for every \(i \in S\).

q5) Which of the following two assertions is correct?

□ Without any further assumption, we know that \(X\) is positive-recurrent.
□ If \(S\) is finite, then we know that \(X\) is positive-recurrent, but in the general case, we do not know.

q6) Which of the following two assertions is correct?

□ Without any further assumption, we know that \(\pi\) is also a limiting distribution.
□ If \(\pi\) satisfies the detailed balance equation, then we know that \(\pi\) is also a limiting distribution, but in the general case, we do not know.
Exercise 1. (30 points)
Let us consider the Markov chain \((X_n, n \geq 0)\) with state space \(S = \{0, 1, 2\}\) and transition graph:

where the parameters \(p, q\) both satisfy \(0 \leq p, q \leq 1\).

a1) Compute the set

\[ D_1 = \{(p, q) \in [0, 1]^2 : \text{the chain } X \text{ with parameters } p, q \text{ is irreducible}\} \]

a2) Compute the set

\[ D_2 = \{(p, q) \in [0, 1]^2 : \text{the chain } X \text{ with parameters } p, q \text{ is both irreducible and aperiodic}\} \]

a3) Explain why the chain \(X\) is ergodic when \((p, q) \in D_2\).

From now on, let us assume that the couple of parameters \((p, q)\) belongs to the set \(D_2\).

b1) Compute the stationary distribution \(\pi\) of the chain.

b2) Is the stationary distribution \(\pi\) also a limiting distribution for all \((p, q) \in D_2\)?

b3) Is the detailed balance equation satisfied for all \((p, q) \in D_2\)?

c1) Compute the eigenvalues \((\lambda_0, \lambda_1, \lambda_2)\) of the transition matrix \(P\) of the chain.

Hint: In order to simplify computations, it might help here to define \(r = 1 - q\) and compute everything in terms of the parameters \((p, r)\) rather than \((p, q)\), and then translate back the expressions obtained in terms of the parameters \((p, q)\).

c2) Compute the corresponding spectral gap \(\gamma\). You may leave the expression for \(\gamma\) in the form \(\gamma = \min(\ldots, \ldots)\) or \(\gamma = \max(\ldots, \ldots)\).

d1) Compute the set

\[ D_3 = \{(p, q) \in D_2 : \text{the stationary distribution } \pi \text{ with parameters } (p, q) \text{ is uniform on } S\} \]

d2) For what values of the parameters \((p, q) \in D_3\) is the spectral gap \(\gamma\) the largest?
Exercise 2. (21 points)

On the state space $S = \{0, \ldots, N\}$ with $N \geq 2$, we would like to use the Metropolis algorithm in order to sample from the distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{N-1}, \pi_N)$, where we assume that

$$\pi_0 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_{N-1} \geq \pi_N > 0$$

We start from the base chain with transition matrix $\Psi$ and corresponding transition graph

![Transition Graph](image)

a1) Establish the list of hypotheses satisfied by this base chain that guarantee the convergence of the Metropolis algorithm.

a2) Compute the transition matrix $P$ of the corresponding Metropolis chain. Do not forget to compute the values of the diagonal elements $p_{ii}$ for $0 \leq i \leq N$.

Let us further assume that for $0 \leq i \leq N$,

$$\pi_i = \frac{c^i}{Z} \quad \text{where} \quad 0 < c < 1 \quad \text{and} \quad Z = \sum_{i=0}^{N} c^i = \frac{1 - c^{N+1}}{1 - c} \quad (1)$$

a3) Compute the transition matrix $P$ in this particular case.

Still under the assumption that the distribution $\pi$ is given by formula (1), imagine now that we start the Metropolis algorithm from the following modified base chain with transition matrix $\tilde{\Psi}$ and corresponding transition graph

![Transition Graph](image)

b1) Are the hypotheses listed in question a1) still satisfied here?

b2) Compute the transition matrix $\tilde{P}$ of the corresponding Metropolis chain.

b3) For what value of $0 < c < 1$ does the Metropolis algorithm introduces the least number of self-loops in the chain?