Let $H$ be an $n \times n$ matrix whose entries are given by

$$H_{jk} = \frac{\exp(i\phi_{jk})}{\sqrt{j+k}}$$

where $\phi_{jk}$ are i.i.d. random phases uniformly distributed on $[0, 2\pi]$.

Let also $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ be the singular values of $H$.

It can be shown, using the classical moments method, that

$$\mathbb{E}(\sigma_1) \leq 2 \sqrt{\log n}$$

By an easy argument (namely, using the fact that $\sigma_1 \geq \sqrt{(HH^*)_{11}}$), it can also be shown that the order of this upper bound is correct, but the constant 2 seems to be loose, according to numerical simulations.

**Open problem 0.1.** Can one say something more precise about the asymptotic behaviour of $\mathbb{E}(\sigma_1)$ or even $\sigma_1$ itself (almost sure result)?

Moreover, the limiting distribution of the singular values seems to converge to a nice distribution in the large $n$ limit, as the following figure shows.

![Histogram of the singular values of $H$ for $n = 1000$.](image)

It can be shown that the Stieltjes transform of the asymptotic eigenvalue distribution of $HH^*$ is given by $g(z) = \int_0^1 g(x, z) \, dx$, where $g(x, z)$ is the solution of the following integral equation:

$$g(x, z) = 1 / \left( -z + \int_0^1 \frac{1}{x+y} g(y, z) \, dy \right), \quad x \in [0, 1], \ z \in \mathbb{C} : \text{Im } z > 0$$

**Open problem 0.2.** Can one characterize the solution of this integral equation and the corresponding eigenvalue distribution?