Solutions 6

1*. a) The sequence $Y_n$ does not admit a limit in distribution. To see this, observe that

$$\text{Var}(Y_n) = \sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{n} \frac{1}{i} \simeq \ln(n) \to +\infty$$

So its limit would have an infinite variance. This, strictly speaking, is not a definitive argument against convergence in distribution (but is accepted if it is the one you gave); a more subtle and definitive argument will follow after part b).

b) The sequence $W_n$ does converge in distribution towards a Gaussian random variable $Z \sim \mathcal{N}(0, \ln(2))$. To prove this, let us consider the sequence of random variables $Z_n \sim \mathcal{N}(0, \frac{1}{n})$, as well as $V_n = Z_{n+1} + \ldots + Z_{2n}$. Observe that

$$\mathbb{E}(X_k) = \mathbb{E}(Z_k) = 0 \quad \text{and} \quad \text{Var}(X_k) = \text{Var}(Z_k) = \frac{1}{k}, \quad \forall k \geq 1$$

Therefore, according to the lemma seen in class, for any function $g \in C^3_b(\mathbb{R})$ with $\sup_{x \in \mathbb{R}} |g'''(x)| \leq C$, we have

$$|\mathbb{E}(g(W_n)) - \mathbb{E}(g(V_n))| \leq \frac{C}{6} \sum_{k=n+1}^{2n} \left( \mathbb{E}(|X_k|^3) + \mathbb{E}(|Z_k|^3) \right)$$

One easily sees that $\mathbb{E}(|X_k|^3) + \mathbb{E}(|Z_k|^3) = O(k^{-3/2})$, so there exists $\tilde{C}$ such that

$$|\mathbb{E}(g(W_n)) - \mathbb{E}(g(V_n))| \leq \tilde{C} \sum_{k=n+1}^{2n} k^{-3/2} \overset{(a)}{\simeq} \tilde{C} \int_{n}^{2n} dx x^{-3/2} = 2\tilde{C} \left( n^{-1/2} - (2n)^{-1/2} \right) \to 0$$

where (a) follows from the first hint given in the problem set. By the Portmanteau theorem, we conclude that the sequences $W_n$ and $V_n$ must converge to the same limit in distribution (if they indeed converge). Observe now that $V_n$, being a sum of independent Gaussians, is itself Gaussian, with mean 0 and variance

$$\text{Var}(V_n) = \sum_{k=n+1}^{2n} \text{Var}(Z_k) = \sum_{k=n+1}^{2n} \frac{1}{k} \simeq \int_{n}^{2n} dx \frac{1}{x} = \ln(2n) - \ln(n) = \ln(2)$$

where the approximation (which becomes more and more accurate as $n$ increases) again follows from the first hint. $V_n$ is therefore a sequence of Gaussian random variables with mean 0 and variance converging towards $\ln(2)$. By the second hint, the result follows.

Back to part a): For $n = 2^k$, we have the following relation between the $Y$’s and the $W$’s:

$$Y_{2^k} = X_1 + X_2 + (X_3 + X_4) + (X_5 + \ldots + X_8) + \ldots + (X_{2^{k-1}+1} + \ldots + X_{2^k}) = X_1 + \sum_{j=0}^{k-1} W_{2^j}$$

From a), we know that as $j \to \infty$, $W_{2^j}$ converges in distribution towards a Gaussian with fixed variance $\ln(2)$. Observe also that all the random variables $W_{2^j}$ are independent. This means that the sequence of random variables $(Y_{2^k}, k \geq 1)$ cannot converge in distribution. As it is a subsequence of the original sequence, the conclusion follows.
2. a) \( \phi_X(t) = \sum_{k=0}^{n} e^{ikt} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k (1-p)^{n-k} = (pe^{it} + 1-p)^n. \)

b) \( \phi_X(t) = \sum_{k \geq 1} e^{ikt} (1-p)^{k-1} p = pe^{it} \sum_{k \geq 0} ((1-p)e^{it})^k = \frac{pe^{it}}{1-(1-p)e^{it}}. \)

c) \[
\phi_X(t) = \int_{\mathbb{R}} e^{itx} e^{-\lambda |x|} dx = \int_{-\infty}^{0} \frac{\lambda}{2} e^{(it+\lambda)x} dx + \int_{0}^{\infty} \frac{\lambda}{2} e^{(it-\lambda)x} dx \\
= \frac{\lambda}{2} \frac{1}{it+\lambda} (1-0) + \frac{\lambda}{2} \frac{1}{it-\lambda} (0-1) = \frac{\lambda^2}{\lambda^2 + t^2}.
\]

d*) We use here the residue theorem of complex analysis. The function \( z \mapsto g_t(z) = e^{itz} \frac{\lambda}{\pi(\lambda^2 + z^2)} \) has two poles of degree one in the complex plane, namely \( +i\lambda \) and \( -i\lambda \). It can be shown that
\[
\phi_X(t) = \int_{\mathbb{R}} g_t(x) dx = \lim_{R \to \infty} \int_{C_R} g_t(z) dz
\]
where the path \( C_R \) is the union of the line segment \([-R,R]\) and the upper semicircle of radius \( R \) (the integral vanishes on this semi-circle as \( R \to \infty \)). By the residue theorem, this integral is given by
\[
\phi_X(t) = 2\pi i \text{Res}(g_t, i\lambda) = \exp(-\lambda |t|).
\]
As a matter of fact, this example is explained in full detail in the following wikipedia entry:


NB: Notice the (approximate) duality between examples c) and d).

3. a) \( X_1 \) admits a pdf, as \( \int_{\mathbb{R}} |\phi_X_1(t)| dt < \infty \). Using the inversion formula, we get
\[
p_{X_1}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} e^{-\lambda |t|} dt = \frac{1}{2\pi} \left( \int_{-\infty}^{0} e^{(\lambda-ix)t} dt + \int_{0}^{\infty} e^{-(\lambda+ix)t} dt \right) \\
= \frac{1}{2\pi} \left( \frac{1}{\lambda - ix} + \frac{1}{\lambda + ix} \right) = \frac{\lambda}{\pi(\lambda^2 + x^2)}.
\]

b) By integration, we obtain
\[
P(|X_1| \leq \lambda) = \frac{1}{\pi} \arctan \left( \frac{t}{\lambda} \right) \bigg|_{-\lambda}^{\lambda} = \frac{1}{\pi} \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \frac{1}{2}.
\]

c) Using the fact that the \( X_j \) are i.i.d., we obtain
\[
\phi_{S_n/n}(t) = \mathbb{E}(\exp(itS_n/n)) = (\mathbb{E}(itX_1/n))^n = (\exp(-\lambda |t/n|))^n = \exp(-\lambda |t|).
\]

d) For all \( n \geq 1 \), \( S_n/n \) therefore has the same characteristic function as \( X_1 \), thus the same distribution, by the inversion formula. We obtain trivially that
\[
\frac{S_n}{n} \xrightarrow{d} X_1.
\]
As trivial as this result is, it is a surprising one! Note that by the law of large numbers, one would expect $S_n/n$ to converge towards a constant. But of course, the law of large numbers does not apply here, as Cauchy random variables are not integrable.

\[ e^*) \quad S_n/n \text{ does not converge in probability towards } X_1. \] One way to see this is the following:

\[
\frac{S_{2n}}{2n} = \frac{1}{2} \left( \frac{S_n}{n} + \frac{\tilde{S}_n}{n} \right),
\]

where $\tilde{S}_n = X_{n+1} + \ldots + X_{2n}$. Notice that by the previous computations, both $\frac{S_n}{n}$ and $\frac{\tilde{S}_n}{n}$ are Cauchy-distributed. Moreover, they are independent. In addition,

\[
\frac{S_{2n}}{2n} - \frac{S_n}{n} = \frac{1}{2} \left( \frac{\tilde{S}_n}{n} - \frac{S_n}{n} \right).
\]

Let us rename now $\frac{\tilde{S}_n}{n} = Y_n$ and $\frac{S_n}{n} = Z_n$. These are two independent Cauchy random variables, so again by the previous computations, $W_n = \frac{1}{2} (Y_n - Z_n)$ is a Cauchy random variable (the minus sign does not change anything). What we have therefore shown is that

\[
\frac{S_{2n}}{2n} - \frac{S_n}{n} = W_n.
\]

This implies that for a given $\varepsilon > 0$,

\[
P\left( \left\{ \left| \frac{S_{2n}}{2n} - \frac{S_n}{n} \right| > \varepsilon \right\} \right) = P(\{|W_n| \geq \varepsilon\})
\]

does not converge to zero as $n$ goes to infinity, as the distribution of $W_n$, which is the Cauchy distribution, does not depend on $n$. But if $\frac{S_n}{n}$ where to converge in probability to some random variable $Z$, then the above expression should converge to zero for all $\varepsilon > 0$ (we use here the fact that if a sequence converges, then it must be a Cauchy sequence; please pay attention that the name of Cauchy appears here twice in different contexts...).