Solutions 2

1. a) use $B = A \cup (B \setminus A)$, where $A$ and $B \setminus A$ are disjoint.

b) use $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, where $B_n = A_n \setminus (A_1 \cup \ldots \cup A_{n-1})$; the $B_n$ are disjoint, so by axiom (ii) and a),

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

c) use b) directly.

d) use again $B = A \cup (B \setminus A)$.

e) use $\Omega = A \cup A^c$ and $P(\Omega) = 1$.

f) use $A \cup B = A \cup (B \setminus (A \cap B))$, where $A$ and $B \setminus (A \cap B)$ are disjoint, along with c).

g) $P(\bigcup_{n \geq 1} A_n) = P(\bigcup_{n \geq 1} (A_n \cap (\bigcup_{i=1}^{n-1} A_i)^c)) = P(\bigcup_{n \geq 1} (A_n \cap A_{n-1}^c)) = \sum_{n=1}^{\infty} P(A_n \cap A_{n-1}^c)$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P(A_i \cap A_{i-1}^c) = \lim_{n \to \infty} P(\bigcup_{i=1}^{n} (A_i \cap A_{i-1}^c)) = \lim_{n \to \infty} P(\bigcup_{i=1}^{n} A_i) = \lim_{n \to \infty} P(A_n).$$

h) $P(\bigcap_{n \geq 1} A_n) = 1 - P((\bigcap_{n \geq 1} A_n)^c) = 1 - P(\bigcup_{n \geq 1} A_n^c) = 1 - \lim_{n \to \infty} P(A_n^c) = \lim_{n \to \infty} P(A_n)$. 

2. a) We have $P(i) = 0.25$ for all $i$, so $P(\{\text{red}\}) = P(\{1, 4\}) = 0.5$, $P(\{\text{odd}\}) = P(\{1, 3\}) = 0.5$, $P(\{1 \text{ or } 2\}) = P(\{1, 2\}) = 0.5$, as well as

$$P(\{\text{red}\} \cap \{\text{odd}\}) = P(\{1\}) = 0.25 = P(\{\text{red}\}) P(\{\text{odd}\}),$$

$$P(\{\text{red}\} \cap \{1 \text{ or } 2\}) = P(\{1\}) = 0.25 = P(\{\text{red}\}) P(\{1 \text{ or } 2\}),$$

$$P(\{\text{odd}\} \cap \{1 \text{ or } 2\}) = P(\{1\}) = 0.25 = P(\{\text{odd}\}) P(\{1 \text{ or } 2\}).$$

So “red”, “odd” and “1 or 2” are 2-by-2 independent, but they are not independent as a family of 3 events, since

$$P(\{\text{red}\} \cap \{\text{odd}\} \cap \{1 \text{ or } 2\}) = P(\{1\}) = 0.25$$

$\neq 0.125 = P(\{\text{red}\}) P(\{\text{odd}\}) P(\{1 \text{ or } 2\}).$

b) In this case, $P(\{\text{red}\}) = 0.5$, $P(\{\text{odd}\}) = 0.5$, $P(\{1 \text{ or } 2\}) = 0.6$, and

$$P(\{\text{red}\} \cap \{\text{odd}\}) = P(\{1\}) = 0.3 \neq P(\{\text{red}\}) P(\{\text{odd}\}),$$

$$P(\{\text{red}\} \cap \{1 \text{ or } 2\}) = P(\{1\}) = 0.3 = P(\{\text{red}\}) P(\{1 \text{ or } 2\}),$$

$$P(\{\text{odd}\} \cap \{1 \text{ or } 2\}) = P(\{1\}) = 0.3 = P(\{\text{odd}\}) P(\{1 \text{ or } 2\}).$$

So “red” and “1 or 2”, as well as “odd” and “1 or 2” are independent, but not “red” and “odd”. From this, one deduces that the family “red”, “odd” and “1 or 2” cannot be independent.
3. a) No. Even though it is easily shown that $Y$ and $Z$ are uncorrelated random variables (i.e. that their covariance is zero), they are not independent. Here is a counter-example: $P(\{Y = +2\}) = P(\{Z = +2\}) = 1/4$, but $P(\{Y = +2, Z = +2\}) = 0$. So we have found two Borel sets $B_1 = \{+2\}$ and $B_2 = \{+2\}$ such that

$$P(\{Y \in B_1, Z \in B_2\}) \neq P(\{Y \in B_1\}) \cdot P(\{Z \in B_2\}).$$

b) Yes. In this case again, $Y$ and $Z$ are uncorrelated. Let us now compute their joint pdf: the joint pdf of $X_1$ and $X_2$ is given by

$$p_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right),$$

Making now the change of variables $y = x_1 + x_2$, $z = x_1 - x_2$, we obtain that

$$x_1^2 + x_2^2 = \left(\frac{y + z}{2}\right)^2 + \left(\frac{y - z}{2}\right)^2 = \frac{y^2 + z^2}{2},$$

so

$$\frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) dx_1 dx_2 = \frac{1}{4\pi} \exp\left(-\frac{y^2 + z^2}{4}\right) dydz.$$

Therefore:

$$p_{Y,Z}(y, z) = \frac{1}{4\pi} \exp\left(-\frac{y^2 + z^2}{4}\right),$$

from which we deduce that $Y$ and $Z$ are independent $\mathcal{N}(0, 2)$ random variables.

4. a) In this case,

$$P^{(1)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1) \cdot \mu(B_2) = P^{(1)}(\{X_1 \in B_1\}) \cdot P^{(1)}(\{X_2 \in B_2\}).$$

The random variables $X_1$ and $X_2$ are therefore independent and identically distributed (i.i.d.).

b) In this case,

$$P^{(2)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1 \cap B_2).$$

Notice first that whenever $B_1 \cap B_2 = \emptyset$, the above probability is zero, so it can never be that $X_1, X_2$ take values simultaneously in disjoint sets $B_1, B_2$. As this holds for any disjoint sets $B_1, B_2$, a little reflection about this shows that $X_1, X_2$ have no choice but to be the same random variable with distribution $\mu$ (up to a set of measure zero). That is, $X_1(\omega) = X_2(\omega)$ for almost all $\omega \in \Omega$.

NB: Please notice that in both cases, the two random variables $X_1, X_2$ have the same distribution, but in one case, they are independent, while in the other, they are the same random variable.