Solutions to Homework 12

Exercise 1. a) Yes: \( x \mapsto x^4 \) is convex, so we conclude by Jensen’s inequality and the fact that \((S_n, n \geq 0)\) is a martingale.

b) Yes: \( \mathbb{E}(S_{n+1}^4 | \mathcal{F}_n) = S_n^4 + 6S_n^2 + 1 \), so \( \mathbb{E}(S_{n+1}^4 - (n + 1) | \mathcal{F}_n) = (S_n^4 - n) + 6S_n^2 \geq S_n^4 - n \).

c) From b), \( \mathbb{E}(S_{n+1}^4) = \mathbb{E}(S_n^4) + 6n + 1 \), so by induction, \( \mathbb{E}(S_n^4) = 3n^2 - 2n \).

d) \( \lim_{n \to \infty} \frac{\mathbb{E}(S_n^4)}{n^2} = 3 \). This could also be deduced directly from the central limit theorem, which states that \( S_n / \sqrt{n} \) converges in distribution to an \( \mathcal{N}(0, 1) \) random variable, whose fourth moment is equal to 3.

Exercise 2. a) The process is a martingale, as i) \( \mathbb{E}(|Y_n|) = \mathbb{E}(Y_n) \leq (3/2)^n \) for all \( n \); ii) \( Y_n \) is \( \mathcal{F}_n \)-measurable for all \( n \), by definition, and
\[
\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \frac{1}{2} \left( \frac{3}{2} Y_n + \frac{1}{2} Y_n \right) = Y_n
\]

b) \( \mathbb{E}(Y_n) = \mathbb{E}(Y_0) = 1 \), since \( Y \) is a martingale and
\[
\mathbb{E}(Y_{n+1}^2) = \mathbb{E}(\mathbb{E}(Y_{n+1}^2|\mathcal{F}_n)) = \mathbb{E} \left( \frac{1}{2} \left( \frac{9}{4} Y_n^2 + \frac{1}{4} Y_n^2 \right) \right) = \frac{5}{4} \mathbb{E}(Y_n^2)
\]
so by induction, \( \mathbb{E}(Y_n^2) = (5/4)^n \) (as \( \mathbb{E}(Y_0^2) = 1 \)) and \( \text{Var}(Y_n) = (5/4)^n - 1 \).

c) The process \( Y \) is not confined to a bounded interval; it can take values between 0 and \( \infty \).

d) As \( \sup_{n \in \mathbb{N}} \mathbb{E}(|Y_n|) = \sup_{n \in \mathbb{N}} \mathbb{E}(Y_n) = 1 < \infty \), the second version of the martingale convergence theorem tells us that there exists a random variable \( Y_\infty \) such that \( Y_n \to Y_\infty \).

e) Notice that
\[
Z_{n+1} = \begin{cases} 
Z_n + \log(3/2), & \text{w.p. } 1/2 \\
Z_n + \log(1/2), & \text{w.p. } 1/2 
\end{cases}
\]
so \( \mathbb{E}(Z_n) = n (\log(3/2) + \log(1/2))/2 = 3/2 \log(3/4) \). As \( \log(3/4) < 0 \), \( Z \) is a random walk with a negative drift. It can be shown that for any \( K > 0 \), there exists \( c > 0 \) such that
\[
\mathbb{P}(Z_n \geq -K) \leq \exp(-cn), \quad \forall n
\]
so \( Z_n \to -\infty \) a.s., implying that \( Y_n = \exp(Z_n) \to 0 = Y_\infty \) a.s.

f) The answer is no, as for every \( n \), \( \mathbb{E}(Y_\infty|\mathcal{F}_n) = \mathbb{E}(0|\mathcal{F}_n) = 0 \neq Y_n \).
Coding Exercise 3. a) Let us compute

\[ \mathbb{E}(G_n^2) = \mathbb{E}\left( \left( \sum_{j=1}^{n} H_j (M_j - M_{j-1}) \right)^2 \right) \]

\[ = \sum_{j=1}^{n} \mathbb{E}(H_j^2 (M_j - M_{j-1})^2) + 2 \sum_{j=1}^{n} \sum_{k>j}^{n} \mathbb{E}(H_j H_k (M_j - M_{j-1}) (M_k - M_{k-1})) \]

All the terms in the double sum on the right-hand side are equal to 0 for the following reason: for \( j < k \),

\[ \mathbb{E}(H_j H_k (M_j - M_{j-1}) (M_k - M_{k-1})) = \mathbb{E}(\mathbb{E}(H_j H_k (M_j - M_{j-1}) (M_k - M_{k-1}) | \mathcal{F}_{k-1})) = \mathbb{E}(H_j H_k (M_j - M_{j-1}) \mathbb{E}(M_k - M_{k-1} | \mathcal{F}_{k-1})) = 0 \]

For the terms in the simple sum, we have

\[ \mathbb{E}(H_j^2 (M_j - M_{j-1})^2) = \mathbb{E}(\mathbb{E}(H_j^2 (M_j - M_{j-1})^2 | \mathcal{F}_{j-1})) = \mathbb{E}(H_j^2 \mathbb{E}((M_j - M_{j-1})^2 | \mathcal{F}_{j-1})) \]

Observe now that

\[ \mathbb{E}((M_j - M_{j-1})^2 | \mathcal{F}_{j-1}) = \mathbb{E}(M_j^2 | \mathcal{F}_{j-1}) - 2M_{j-1} \mathbb{E}(M_j | \mathcal{F}_{j-1}) + M_{j-1}^2 \]

\[ = \mathbb{E}(M_j^2 | \mathcal{F}_{j-1}) - M_{j-1}^2 = A_j - A_{j-1} \]

by definition of the process \( A \) such that \( M^2 - A \) is a martingale. We therefore finally obtain:

\[ \mathbb{E}(G_n^2) = \sum_{j=1}^{n} \mathbb{E}(H_j^2 (A_j - A_{j-1})) \]

b) A sufficient condition for \( G_{\infty} \) to exist is

\[ \sup_{n \in \mathbb{N}} \mathbb{E}(G_n^2) < +\infty \]

For the simple symmetric random walk \( S \), we have seen that \( A_n = n \), so the above condition becomes

\[ \sup_{n \in \mathbb{N}} \sum_{j=1}^{n} \mathbb{E}(H_j^2) = \sum_{n \geq 1} \mathbb{E}(H_n^2) < +\infty \] (1)

c) \( H^{(1)} \) and \( H^{(2)} \) satisfy the summability condition (1), so the corresponding processes \( G \) converge almost surely, as seen on the two graphs below:
d) On the contrary, $H^{(3)}$ and $H^{(5)}$ do not satisfy the summability condition (1), so \(^1\) the corresponding processes $G$ do not converge almost surely, as seen on the two graphs below:

![Graphs](image1.png)  

$G^{(3)}$  $G^{(5)}$

We observe that the fluctuations of $G$ are not so large as $n$ increases. This is related to the fact that in both cases, $\sum_{n=1}^{N} \mathbb{E}(H_n^2) \simeq \log(N)$ grows slowly to $+\infty$.

Finally, the process $H^{(4)}$ is not even predictable, so $G^{(4)}$ is not a martingale, as can be seen below:

![Graph](image2.png)  

$G^{(4)}$

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\(^1\)Strictly speaking, the fact that condition (1) is not satisfied does not imply that $G$ does not converge. But here, we observe that it so happens.