Matlab Solutions 1

1. a) \[
\mathbb{E}(S_N) = \sum_{n=1}^{N} \mathbb{E}(X_n) = N(2p - 1) \quad \text{and} \quad \text{Var}(S_N) = \sum_{n=1}^{N} \text{Var}(X_n) = 4Np(1-p)
\]

b) Numerically, we observe that $\hat{\mu}_N \approx 0$ and $\hat{\sigma}_N \approx \sqrt{N}$. See the matlab code regarding the histograms: as $N$ gets large, the shape looks more and more like a Gaussian, with increasing variance around 0.

c) Numerically, we observe that $\hat{\mu}_N \approx \frac{N}{3}$ and $\hat{\sigma}_N \approx \sqrt{\frac{8N}{9}}$. See the matlab code regarding the histograms: as $N$ gets large, the shape looks also more and more like a Gaussian, although the distribution gets more and more shifted away from 0, so that seen from 0, the distribution looks like highly peaked close to the value $N/3$ (even though it has more or less the same spread as in the case $p = 1/2$). In particular, we observe that the probability that the random walk hits the value 0 decreases rapidly as $N$ increases.

d) See the matlab code regarding the histograms: independently of the value of $p$, the histogram of the random variables $Z_{N}^{(1)}, \ldots, Z_{N}^{(M)}$ approaches quickly that of a $\mathcal{N}(0,1)$ distribution as $N$ grows. This is a visualization of the central limit theorem (to come in Lecture 6).

2. a) As $M$ grows, one observes more and more fluctuations of the mean and the standard deviation. The fact is, one can prove theoretically that $\mathbb{E}(T) = +\infty$ and that $\text{Var}(T)$ is ill-defined. So one does not expect the simulations to give anything interesting as the number of samples $M$ increases.

b) Even though the mean is infinite and the variance does not exist, the random variable $T$ has a distribution that can be computed theoretically (whose main feature is that $\mathbb{P}(\{T > n\}) \approx \frac{1}{\sqrt{n}}$, which explains the fact that the expectation is infinite). See the code regarding the histograms. As $M$ increases, one indeed observes that the histogram stabilizes somehow: still, the distribution of $T$ is heavy-tailed, which means that $T$ takes very large values from time to time, and this creates large fluctuations of the histogram from one code execution to the next. Also, if one does not let the random walk run long enough, then one observes that it is often the case that the random walk never comes back to 0: this creates an artificial peak on large values of $T$.

c) In this case also, the mean of $T$ is infinite and its variance is ill-defined, so we encounter the same problems as in part a). But now, drawing the histogram becomes much more difficult, as it is often the case that the random walk never comes back to 0, even after a very long time. One can actually prove in this case that $\mathbb{P}(\{T = +\infty\}) > 0$. 


3. In order for a random variable to be measurable with respect to the \( \sigma \)-field \( \mathcal{F} \), it should be constant on all the atoms of \( \mathcal{F} \), which are sets of the form

\[
A_1^* \cap A_2^* \cap \ldots \cap A_n^* = \{ \omega \in \Omega : \omega_1 \in [0.5, 1]^*, \omega_2 \in [0.5, 1]^*, \ldots, \omega_n \in [0.5, 1]^* \}
\]

where \( A_i^* \) is either \( A_i \) or \( A_i^c \) (resp., \([0.5, 1]^* \) is either \([0.5, 1] \) or \([0, 0.5] \)).

a) \( X(\omega) = \sum_{i=1}^n \omega_i \) does clearly not meet the above requirement.

b) \( Y(\omega) = \# \{ 1 \leq i \leq n : \omega_i \in [0.5, 1] \} = \sum_{i=1}^n 1_{A_i}(\omega) \), so \( Y \) is, by definition, measurable with respect to \( \mathcal{F} \).

c) \( Z(\omega) = \sum_{i=1}^n [2\omega_i - 1] = Y(\omega) \), as \([2\omega_i - 1] = 1_{\{\omega_i \in [0.5, 1]\}} \). So \( Z \) is also \( \mathcal{F} \)-measurable.

d) \( W(\omega) = \lceil \sum_{i=1}^n (2\omega_i - 1) \rceil \) does not meet the above requirement. For example, for \( n = 2 \), we have

\[
W(\omega) = \lceil 2(\omega_1 + \omega_2) - 2 \rceil
\]

whose value is not constant over \( \{ \omega \in \Omega : \omega_1 \in [0, 0.5], \omega_2 \in [0, 0.5] \} \). E.g., it takes the value \(-1\) for \((\omega_1, \omega_2)\) close to \((0, 0)\) and takes the value \(0\) for \((\omega_1, \omega_2)\) close to \((0.5, 0.5)\).

4. a) Using independence (together with the fact that the random variables are bounded, therefore integrable, square-integrable and more), we obtain:

\[
\text{Cov}(XY, XZ) = \mathbb{E}(XYXZ) - \mathbb{E}(XY)\mathbb{E}(XZ) = \mathbb{E}(X^2)\mathbb{E}(Y)\mathbb{E}(Z) - \mathbb{E}(X)^2\mathbb{E}(Y)\mathbb{E}(Z)
\]

and

\[
\text{Var}(XY) = \mathbb{E}(XYXY) - \mathbb{E}(XY)^2 = \mathbb{E}(X^2)\mathbb{E}(Y^2) - \mathbb{E}(X)^2\mathbb{E}(Y)^2 = \mathbb{E}(X)^2\mathbb{E}(Y^2) + \mathbb{E}(X)^2\text{Var}(Y)
\]

\[
= \mathbb{E}(X)\text{Var}(Y) + \mathbb{E}(X)^2\text{Var}(Y)
\]

b) Here, one has to take into account the possible dependency between \( Y \) and \( Z \):

\[
\text{Cov}(XY, XZ) = \mathbb{E}(XYXZ) - \mathbb{E}(XY)\mathbb{E}(XZ) = \mathbb{E}(X^2)\mathbb{E}(YZ) - \mathbb{E}(X)^2\mathbb{E}(Y)\mathbb{E}(Z)
\]

\[
= \mathbb{E}(X^2)\text{Cov}(Y, Z) + \text{Var}(X)\mathbb{E}(Y)\mathbb{E}(Z)
\]

\[
= \text{Var}(X)\text{Cov}(Y, Z) + \mathbb{E}(X)^2\text{Cov}(Y, Z) + \text{Var}(X)\mathbb{E}(Y)\mathbb{E}(Z)
\]

Observe that we recover the formula obtained in a) when \( Y \) and \( Z \) are independent (but uncorrelated would also do).