Midterm Exam

SURNAME: ........................ FIRST NAME: ........................ SECTION: ............

PLEASE JUSTIFY ALL YOUR ANSWERS !!!

Exercise 1. (10 points)

Reminder for this exercise: \( \frac{1}{\ln(n)} \sum_{i=1}^{n} \frac{1}{i} \rightarrow 1 \) and \( \sum_{i=1}^{n} \frac{1}{i^2} \rightarrow \frac{\pi^2}{6} \)

Let \( (X_n, n \geq 1) \) be a sequence of square-integrable and independent random variables such that

\[ \mathbb{E}(X_n) = \frac{1}{n}, \quad \text{Var}(X_n) = \frac{1}{n^2}, \quad n \geq 1 \]

Let us also define \( S_n = X_1 + \ldots + X_n \) for \( n \geq 1 \).

a) Compute \( \mathbb{E}(S_n) \) and \( \text{Var}(S_n) \).

b) Does there exist a sequence of real numbers \( (\mu_n, n \geq 1) \) such that

\[ \frac{S_n - \mu_n}{n} \xrightarrow{p} 0 \quad n \rightarrow \infty \]

If yes, prove it; if no, explain why.

c) Does it also hold that

\[ \frac{S_n - \mu_n}{n} \xrightarrow{a.s.} 0 \quad n \rightarrow \infty \]

Justify your answer!

Assume now that all the random variables \( X_n \) are non-negative.

d) Prove that for every fixed \( t > 0 \),

\[ \lim_{n \rightarrow \infty} \mathbb{P}(\{S_n > t\}) > 0 \]

NB: As \( S_n \) is a sum of non-negative random variables, the above sequence of probabilities is increasing, so the limit is guaranteed to exist (for every fixed \( t > 0 \)).

e) Provide an example of distribution such that \( X_n(\omega) \geq 0 \) for every \( \omega \in \Omega \), \( \mathbb{E}(X_n) = \frac{1}{n} \) and \( \text{Var}(X_n) = \frac{1}{n^2} \).

Hint: Look for a random variable \( X_n \) taking only two non-negative values \( a_n, b_n \) with equal probability \( (= 1/2) \).
Exercise 2. (10 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function and $(Z_n, n \geq 1)$ be a sequence of i.i.d. square-integrable random variables. Let also $(X_n, n \geq 0)$ be the sequence of random variables defined recursively as follows:

$$X_0 = 0, \quad X_{n+1} = f(X_n) + Z_{n+1}, \quad n \geq 0$$

a) Prove by induction that for every $n \geq 1$, there exists a Borel-measurable function $f_n : \mathbb{R}^n \to \mathbb{R}$ such that $X_n = f_n(Z_1, Z_2, \ldots, Z_n)$.

b) Deduce that for every $n \geq 0$, $f(X_n)$ and $Z_{n+1}$ are independent.

Assume now that $f(x) = ax$ for some $a \in \mathbb{R}$.

c) Compute recursively $\mathbb{E}(X_n)$ and $\text{Var}(X_n)$.

Hint: For $b \in \mathbb{R}$ with $b \neq 1$, $\sum_{k=0}^{n-1} b^k = \frac{1 - b^n}{1 - b}$.

d) For which values of $a \in \mathbb{R}$ does there exist a constant $K > 0$ such that $\mathbb{E}(X_n^2) \leq K$ for all $n \geq 1$?

e) Show that for the values of $a \in \mathbb{R}$ found in part d), the sequence of distributions of $X_n$ is tight, i.e., that

$$\forall \varepsilon > 0, \quad \exists M > 0 \quad \text{such that} \quad \sup_{n \geq 1} \mathbb{P}(|X_n| \geq M) \leq \varepsilon.$$ 

Assume moreover now that the random variables $Z_n$ are i.i.d. $\sim \mathcal{N}(0, 1)$.

f) For the values of $a \in \mathbb{R}$ found in part d), show that the sequence of random variables $X_n$ converges in distribution and compute the limiting distribution.