Matlab Homework 4 (due Thursday, June 1)

Exercise 1.

Part I. Let \((X_n, n \geq 1)\) be a sequence of i.i.d. random variables such that \(P\{X_n = +1\} = p\) and \(P\{X_n = -1\} = 1 - p\) for some fixed \(0 < p < 1/2\).

Let \(S_0 = 0\) and \(S_n = X_1 + \ldots + X_n, n \geq 1\). Let also \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n), n \geq 1\).

Preliminary question. Deduce from Hoeffding’s inequality that for any \(0 < p < 1/2\),
\[
P\{|S_n - n(2p - 1)| \geq nt\} \leq 2 \exp\left(-\frac{nt^2}{2}\right) \quad \forall t > 0, n \geq 1.
\]
This inequality will be useful at some point in this exercise.

Let now \((Y_n, n \in \mathbb{N})\) be the process defined as \(Y_n = \lambda S_n\) for some \(\lambda > 0\) and \(n \in \mathbb{N}\).

a) Using Jensen’s inequality only, for what values of \(\lambda\) can you conclude that the process \(Y\) is a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)?

b) Identify now the values of \(\lambda > 0\) for which it holds that the process \((Y_n = \lambda S_n, n \in \mathbb{N})\) is a martingale / submartingale / supermartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

c) Compute \(E(|Y_n|)\) and \(E(Y_n^2)\) for every \(n \in \mathbb{N}\) (and every \(\lambda > 0\)).

d) For what values of \(\lambda > 0\) does it hold that \(\sup_{n \in \mathbb{N}} E(|Y_n|) < +\infty?\) \(\sup_{n \in \mathbb{N}} E(Y_n^2) < +\infty?\)

e) Run the process \(Y\) numerically. For what values of \(\lambda > 0\) do you observe that there exists a random variable \(Y_\infty\) such that \(Y_n \xrightarrow{n \to \infty} Y_\infty\) a.s.? Prove it then theoretically and compute the random variable \(Y_\infty\) when it exists (this computation might depend on \(\lambda\), of course).

f) For what values of \(\lambda > 0\) does it hold that \(Y_n \xrightarrow{L^2} Y_\infty?\)

g) Finally, for what values of \(\lambda > 0\) does it hold that \(E(Y_\infty|\mathcal{F}_n) = Y_n, \forall n \in \mathbb{N}\)?
**Part II.** Consider now the (interesting) value $\lambda$ for which the process $Y$ is a martingale. (Spoiler: there is a unique such value of $\lambda$, and it is greater than 1.)

Let $a \geq 1$ be an integer and consider the stopping time $T_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a \text{ or } Y_n \leq \lambda^{-a}\}$.

a) Estimate numerically $P(\{Y_{T_a} = \lambda^a\})$ for some values of $a$. Explain your method.

b) Is it true that $E(Y_{T_a}) = E(Y_0)$? Justify your answer.

c) If possible, use the previous statement to compute $P = P(\{Y_{T_a} = \lambda^a\})$ theoretically. How fast does this probability decay with $a$?

Consider finally the other stopping time $T'_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a\}$.

d) Estimate numerically $P(\{Y_{T'_a} = \lambda^a\})$ for some values of $a$. Explain your method.

e) Is it true that $E(Y_{T'_a}) = E(Y_0)$? Justify your answer.

f) If possible, use the above statement to compute $P' = P(\{Y_{T'_a} = \lambda^a\})$ theoretically. Is this probability $P'$ greater or smaller than $P$?