Advanced Probability and Applications: WEEK 11

1 Martingale transforms

**Definition 1.1.** A process \((H_n, n \in \mathbb{N})\) is said to be *predictable* with respect to a filtration \((\mathcal{F}_n, n \in \mathbb{N})\) if \(H_0 = 0\) and \(H_n\) is \(\mathcal{F}_{n-1}\)-measurable \(\forall n \geq 1\).

**Remark.** If a process is predictable, then it is adapted.

Let now \((\mathcal{F}_n, n \in \mathbb{N})\) be a filtration, \((H_n, n \in \mathbb{N})\) be a predictable process with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) and \((M_n, n \in \mathbb{N})\) be a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

**Definition 1.2.** The process \(G\) defined as

\[
G_0 = 0, \quad G_n = (H \cdot M)_n = \sum_{i=1}^{n} H_i (M_i - M_{i-1}), \quad n \geq 1,
\]

is called the *martingale transform* of \(M\) through \(H\).

**Remark.** This process is the discrete version of the stochastic integral. It represents the gain obtained by applying the strategy \(H\) to the game \(M\):

- \(H_i\) = amount bet on day \(i\) (\(\mathcal{F}_{i-1}\)-measurable).
- \(M_i - M_{i-1}\) = increment of the process \(M\) on day \(i\).
- \(G_n\) = gain on day \(n\).

**Proposition 1.3.** If \(H_n\) is a bounded random variable for each \(n\) (i.e., \(|H_n(\omega)| \leq K_n \; \forall \omega \in \Omega\)), then the process \(G\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

In other words, one cannot win on a martingale!

**Proof.** (i) \(\mathbb{E}(|G_n|) \leq \sum_{i=1}^{n} \mathbb{E}(|H_i| |M_i - M_{i-1}|) \leq \sum_{i=1}^{n} K_i (\mathbb{E}(|M_i|) + \mathbb{E}(|M_{i-1}|)) < \infty\).

(ii) \(G_n\) is \(\mathcal{F}_n\)-measurable by construction.

(iii) \(\mathbb{E}(G_{n+1}|\mathcal{F}_n) = \mathbb{E}(G_n + H_{n+1} (M_{n+1} - M_n)|\mathcal{F}_n) = G_n + H_{n+1} \mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = G_n + 0 = G_n\).

**Example: “the” martingale.**

Let \((M_n, n \in \mathbb{N})\) be the simple symmetric random walk (\(M_n = \xi_1 + \ldots + \xi_n\)) and consider the following strategy:

\[
H_0 = 0, \quad H_1 = 1, \quad H_{n+1} = \begin{cases} 2H_n, & \text{if } \xi_1 = \ldots = \xi_n = +1, \\ 0, & \text{otherwise}. \end{cases}
\]

Notice that all the \(H_n\) are bounded random variables. Then by the above proposition, the process \(G\) defined as

\[
G_0 = 0, \quad G_n = \sum_{i=1}^{n} H_i (M_i - M_{i-1}) = \sum_{i=1}^{n} H_i \xi_i, \quad n \geq 1,
\]

is a martingale. So \(\mathbb{E}(G_n) = \mathbb{E}(G_0) = 0, \forall n \in \mathbb{N}\). Let now

\[
T = \inf\{n \geq 1 : \xi_n = +1\}.
\]

\(T\) is a stopping time and it is easily seen that \(G_T = +1\). But then \(\mathbb{E}(G_T) = 1 \neq 0 = \mathbb{E}(G_0)\)? Is there a contradiction? Actually no. The optional stopping theorem does not apply here, because the time \(T\) is unbounded: \(\mathbb{P}(T = n) = 2^{-n}, \forall n \in \mathbb{N}\), i.e., there does not exist \(N\) fixed such that \(T(\omega) \leq N, \forall \omega \in \Omega\).
2 Doob’s decomposition theorem

**Theorem 2.1.** Let \((X_n, n \in \mathbb{N})\) be a submartingale with respect to a filtration \((\mathcal{F}_n, n \in \mathbb{N})\). Then there exists a martingale \((M_n, n \in \mathbb{N})\) with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) and a process \((A_n, n \in \mathbb{N})\) predictable with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) and increasing (i.e., \(A_n \leq A_{n+1} \forall n \in \mathbb{N}\)) such that \(A_0 = 0\) and \(X_n = M_n + A_n, \forall n \in \mathbb{N}\). Moreover, this decomposition of the process \(X\) is unique.

**Proof.** (main idea) 
\[ \mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n, \] so a natural candidate for the process \(A\) is to set \(A_0 = 0\) and \(A_{n+1} = A_n + \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n (\geq A_n)\), which is a predictable and increasing process. Then, \(M_0 = X_0\) and \(M_{n+1} - M_n = X_{n+1} - X_n - (A_{n+1} - A_n) = X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)\) is indeed a martingale, as \(\mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = 0\).

\[ \square \]

3 The martingale convergence theorem (part I)

3.1 Preliminary: Doob’s martingale

**Proposition 3.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \((\mathcal{F}_n, n \in \mathbb{N})\) be a filtration and \(X: \Omega \rightarrow \mathbb{R}\) be an \(\mathcal{F}\)-measurable and integrable random variable. Then the process \((M_n, n \in \mathbb{N})\) defined as

\[ M_n = \mathbb{E}(X|\mathcal{F}_n), \quad n \in \mathbb{N}, \]

is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

**Proof.** (i) \(\mathbb{E}(|M_n|) = \mathbb{E}(|\mathbb{E}(X|\mathcal{F}_n)|) \leq \mathbb{E}(\mathbb{E}(|X| |\mathcal{F}_n)|)) = \mathbb{E}(|X|) < \infty, \) for all \(n \in \mathbb{N}\).

(ii) By the definition of conditional expectation, \(M_n = \mathbb{E}(X|\mathcal{F}_n)\) is \(\mathcal{F}_n\)-measurable, for all \(n \in \mathbb{N}\).

(iii) \(\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_{n+1}) |\mathcal{F}_n) = \mathbb{E}(X|\mathcal{F}_n) = M_n, \) for all \(n \in \mathbb{N}\). \(\square\)

**Remarks.** - This process describes the situation where one acquires more and more information about a random variable. Think e.g. at the case where \(X\) is a number drawn uniformly at random between 0 and 1, and one reads this number from left to right: while reading, one obtains more and more information about the number.

- Is this a very particular type of martingale? No! As the following paragraph shows, this “example” is actually quite general...

3.2 The martingale convergence theorem: first version

**Theorem 3.2.** Let \((M_n, n \in \mathbb{N})\) be a square-integrable martingale (i.e. a martingale such that \(\mathbb{E}(M_n^2) < \infty\) for all \(n \in \mathbb{N}\)) with respect to a filtration \((\mathcal{F}_n, n \in \mathbb{N})\). Under the additional assumption that

\[ \sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) < \infty, \]

there exists a limiting random variable \(M_\infty\) such that

(i) \(M_n \xrightarrow{n \to \infty} M_\infty\) almost surely.

(ii) \(\lim_{n \to \infty} \mathbb{E}((M_n - M_\infty)^2) = 0\) (quadratic convergence).

(iii) \(M_n = \mathbb{E}(M_\infty |\mathcal{F}_n), \) for all \(n \in \mathbb{N}\) (this last property is referred to as the martingale \(M\) being “closed at infinity”).
Remarks. - Condition (1) is of course much stronger than just asking that $E(M_n^2) < \infty$ for every $n$. Think for example at the simple symmetric random walk $S_n$: $E(S_n^2) = n < \infty$ for every $n$, but the supremum is infinite.

- By conclusion (iii) in the theorem, any square-integrable martingale satisfying condition (1) is actually a Doob martingale (take $X = M_\infty$)!

- A priori, one could think that all the conclusions of the theorem hold true if one replaces all the squares by absolute values in the above statement (such as e.g. replacing condition (1) by $\sup_{n \in \mathbb{N}} E(|M_n|) < \infty$, etc.). This is wrong, and we will see interesting counter-examples later.

- A stronger condition than (1) (leading therefore to the same conclusion) is the following:

$$\sup_{n \in \mathbb{N}} \sup_{\omega \in \Omega} |M_n(\omega)| < \infty.$$  \hfill (2)

Martingales satisfying this stronger condition are called bounded martingales.

Example 3.3. Let $M_0 = x$, where $x \in [0, 1]$ is a fixed number, and let us define recursively:

$$M_{n+1} = \begin{cases} 
M_n^2, & \text{with probability } \frac{1}{2}, \\
2M_n - M_n^2, & \text{with probability } \frac{1}{2}.
\end{cases}$$

The process $M$ is a bounded martingale. Indeed:

(i) By induction, if $M_n \in [0, 1]$, then $M_{n+1} \in [0, 1]$, for every $n \in \mathbb{N}$, so as $M_0 = x \in [0, 1]$, we obtain

$$\sup_{n \in \mathbb{N}, \omega \in \Omega} |M_n(\omega)| \leq 1 < \infty.$$

(ii) $E(M_{n+1}|\mathcal{F}_n) = \frac{1}{2} M_n^2 + \frac{1}{2} (2M_n - M_n^2) = M_n$, for every $n \in \mathbb{N}$.

By the theorem, there exists therefore a random variable $M_\infty$ such that the three conclusions of the theorem hold. In addition, it can be shown by contradiction that $M_\infty$ takes values in the binary set $\{0, 1\}$ only, so that

$$x = E(M_0) = E(M_\infty) = P(M_\infty = 1).$$

3.3 Consequences of the theorem

Before diving into the proof of the above important theorem, let us first explore a few of its interesting consequences.

Optimal stopping theorem, version 2. Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration, let $(M_n, n \in \mathbb{N})$ be a square-integrable martingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ which satisfies condition (1) and let $0 \leq T_1 \leq T_2 \leq \infty$ be two stopping times with respect to $(\mathcal{F}_n, n \in \mathbb{N})$. Then

$$E(M_{T_2}|\mathcal{F}_{T_1}) = M_{T_1} \text{ a.s. and } E(M_{T_2}) = E(M_{T_1}).$$

Proof. Simply replace $N$ by $\infty$ in the proof of the first version and use the fact that $M$ is a closed martingale by the convergence theorem. \hfill \square

Stopped martingale. Let $(M_n, n \in \mathbb{N})$ be a martingale and $T$ be a stopping time with respect to a filtration $(\mathcal{F}_n, n \in \mathbb{N})$, without any further assumption. Let us also define the stopped process

$$(M_{T \wedge n}, n \in \mathbb{N}),$$

where $T \wedge n = \min\{T, n\}$ by definition. Then this stopped process is also a martingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ (we skip the proof here, which uses the first version of the optional stopping theorem).
Optional stopping theorem, version 3. Let \((M_n, n \in \mathbb{N})\) be a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) such that there exists \(c > 0\) with \(|M_{n+1}(\omega) - M_n(\omega)| \leq c\) for all \(\omega \in \Omega\) and \(n \in \mathbb{N}\) (this assumption ensures that the martingale does not make jumps of uncontrolled size: the simple symmetric random walk \(S_n\) satisfies in particular this assumption). Let also \(a, b > 0\) and
\[
T = \inf\{n \in \mathbb{N} : M_n \leq -a \text{ or } M_n \geq b\}.
\]
Observe that \(T\) is a stopping time with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) and that \(-a - c \leq M_T(\omega) \leq b + c\) for all \(\omega \in \Omega\) and \(n \in \mathbb{N}\). In particular,
\[
\sup_{n \in \mathbb{N}} \mathbb{E}(M^2_{T \wedge n}) < \infty,
\]
so the stopped process \((M_{T \wedge n}, n \in \mathbb{N})\) satisfies the assumptions of the first version of the martingale convergence theorem. By the conclusion of this theorem, the stopped martingale \((M_{T \wedge n}, n \in \mathbb{N})\) is closed, i.e. it admits a limit \(M_{T \wedge \infty} = M_T\) and
\[
\mathbb{E}(M_T) = \mathbb{E}(M_{T \wedge \infty}) = \mathbb{E}(M_{T \wedge 0}) = \mathbb{E}(M_0).
\]

Application. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk (which satisfies the above assumptions with \(c = 1\)) and \(T\) be the above stopping time (with \(a, b\) positive integers). Then \(\mathbb{E}(S_T) = \mathbb{E}(S_0) = 0\). Given that \(S_T \in \{-a, +b\}\), we obtain
\[
0 = \mathbb{E}(S_T) = (+b) \mathbb{P}(\{S_T = +b\}) + (-a) \mathbb{P}(\{S_T = -a\}) = bp - a(1 - p), \quad \text{where } p = \mathbb{P}(\{S_T = +b\}),
\]
From this, we deduce that \(\mathbb{P}(\{S_T = +b\}) = p = \frac{a}{a + b}\).

Remark. Note that the same reasoning does not hold if we replace the stopping time \(T\) by a stopping time of the form
\[
T' = \inf\{n \in \mathbb{N} : M_n \geq b\}.
\]
There is indeed no guarantee in this case that the stopped martingale \((M_{T' \wedge n}, n \in \mathbb{N})\) is bounded (from below).