1 Martingales: basic definitions

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

**Definition 1.1.** A filtration is a sequence \((\mathcal{F}_n, n \in \mathbb{N})\) of sub-\(\sigma\)-fields of \(\mathcal{F}\) such that \(\mathcal{F}_n \subset \mathcal{F}_{n+1}, \forall n \in \mathbb{N}\).

**Example.** Let \(\Omega = [0, 1]\), \(\mathcal{F} = \mathcal{B}([0, 1])\), \(X_n(\omega) = n^{th}\) decimal of \(\omega\), for \(n \geq 1\). Let also \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\). Then \(\mathcal{F}_n \subset \mathcal{F}_{n+1}, \forall n \in \mathbb{N}\).

**Definitions 1.2.** - A discrete-time process \((X_n, n \in \mathbb{N})\) is said to be adapted to the filtration \((\mathcal{F}_n, n \in \mathbb{N})\) if \(X_n\) is \(\mathcal{F}_n\)-measurable \(\forall n \in \mathbb{N}\).

- The natural filtration of a process \((X_n, n \in \mathbb{N})\) is defined as \(\mathcal{F}_n^X = \sigma(X_0, \ldots, X_n), n \in \mathbb{N}\). It represents the available amount of information about the process at time \(n\).

**Remark.** A process is adapted to its natural filtration, by definition.

Let now \((\mathcal{F}_n, n \in \mathbb{N})\) be a given filtration.

**Definition 1.3.** A discrete-time process \((M_n, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\) if
(i) \(\mathbb{E}(|M_n|) < \infty, \forall n \in \mathbb{N}\).
(ii) \(M_n\) is \(\mathcal{F}_n\)-measurable, \(\forall n \in \mathbb{N}\) (i.e., \((M_n, n \in \mathbb{N})\) is adapted to \((\mathcal{F}_n, n \in \mathbb{N})\)).
(iii) \(\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n\) a.s., \(\forall n \in \mathbb{N}\).

A martingale is therefore a fair game: the expectation of the process at time \(n + 1\) given the information at time \(n\) is equal to the value of the process at time \(n\).

**Remark.** Conditions (ii) and (iii) are actually redundant, as (iii) implies (ii).

**Properties.** If \((M_n, n \in \mathbb{N})\) is a martingale, then
- \(\mathbb{E}(M_{n+1}) = \mathbb{E}(M_n) (= \ldots = \mathbb{E}(M_0)), \forall n \in \mathbb{N}\) (by the first property of conditional expectation).
- \(\mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = 0\) a.s. (nearly by definition).
- \(\mathbb{E}(M_{n+m}|\mathcal{F}_n) = M_n\) a.s., \(\forall n, m \in \mathbb{N}\).

This last property is important, as it says that the martingale property propagates over time. Here is a short proof, which uses the towering property of conditional expectation:

\[
\mathbb{E}(M_{n+m}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(M_{n+m}|\mathcal{F}_{n+m-1})|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(M_{n+m-1}|\mathcal{F}_n)) = \ldots = \mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n.
\]

**Example: the simple symmetric random walk.**

Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk: \(S_0 = 0, S_n = \xi_1 + \ldots + \xi_n\), where the \(\xi_n\) are i.i.d. and \(\mathbb{P}(\{\xi_1 = +1\}) = \mathbb{P}(\{\xi_1 = -1\}) = 1/2\).

Let us define the following filtration: \(\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \geq 1\). Then \((S_n, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\). Indeed:
(i) \(\mathbb{E}(|S_n|) \leq \mathbb{E}(|\xi_1|) + \ldots + \mathbb{E}(|\xi_n|) = 1 + \ldots + 1 = n < \infty, \forall n \in \mathbb{N}\).
(ii) \(S_n = \xi_1 + \ldots + \xi_n\) is a function of \((\xi_1, \ldots, \xi_n)\), i.e., is \(\sigma(\xi_1, \ldots, \xi_n) = \mathcal{F}_n\)-measurable.
(iii) We have
\[
\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n + \xi_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n|\mathcal{F}_n) + \mathbb{E}(\xi_{n+1}|\mathcal{F}_n) = S_n + \mathbb{E}(\xi_{n+1}) = S_n + 0 = S_n \text{ a.s.}
\]

The first equality on the second line follows from the fact that \(S_n\) is \(\mathcal{F}_n\)-measurable and that \(\xi_{n+1}\) is independent of \(\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)\).
**Generalization.** If the random variables $\xi_n$ are i.i.d. and such that $\mathbb{E}(|\xi_1|) < \infty$ and $\mathbb{E}(\xi_1) = 0$, then $(S_n, n \in \mathbb{N})$ is also a martingale (in particular, $\xi_1 \sim \mathcal{N}(0, 1)$ works).

**Definition 1.4.** Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration. A process $(M_n, n \in \mathbb{N})$ is a submartingale (resp. a supermartingale) with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ if

(i) $\mathbb{E}(|M_n|) < \infty$, $\forall n \in \mathbb{N}$.

(ii) $M_n$ is $\mathcal{F}_n$-measurable, $\forall n \in \mathbb{N}$.

(iii) $\mathbb{E}(M_{n+1}|\mathcal{F}_n) \geq M_n$ a.s., $\forall n \in \mathbb{N}$ (resp. $\mathbb{E}(M_{n+1}|\mathcal{F}_n) \leq M_n$ a.s., $\forall n \in \mathbb{N}$).

**Remarks.** - Not every process is either a sub- or a supermartingale!

- The appellations sub- and supermartingale are counter-intuitive. They are due to historical reasons.

- Condition (ii) is now necessary in itself, as (iii) does not imply it.

- If $(M_n, n \in \mathbb{N})$ is both a submartingale and a supermartingale, then it is a martingale.

**Example:** The simple asymmetric random walk.

- If $\mathbb{P}(\{\xi_1 = +1\}) = p = 1 - \mathbb{P}(\{\xi_1 = -1\})$ with $p \geq 1/2$, then $S_n = \xi_1 + \ldots + \xi_n$ is a submartingale.

- More generally, $S_n = \xi_1 + \ldots + \xi_n$ is a submartingale if $\mathbb{E}(\xi_1) \geq 0$.

**Proposition 1.5.** If $(M_n, n \in \mathbb{N})$ is a martingale with respect to a filtration $(\mathcal{F}_n, n \in \mathbb{N})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable and convex function such that $\mathbb{E}(|\varphi(M_n)|) < \infty$, $\forall n \in \mathbb{N}$, then $(\varphi(M_n), n \in \mathbb{N})$ is a submartingale.

**Proof.** (i) $\mathbb{E}(|\varphi(M_n)|) < \infty$ by assumption.

(ii) $\varphi(M_n)$ is $\mathcal{F}_n$-measurable as $M_n$ is (and $\varphi$ is Borel-measurable).

(iii) $\mathbb{E}(\varphi(M_{n+1})|\mathcal{F}_n) \geq \varphi(\mathbb{E}(M_{n+1}|\mathcal{F}_n)) = \varphi(M_n)$ a.s.

In (iii), the first inequality follows from Jensen’s inequality and the second follows from the fact that $M$ is a martingale.

**Example.** If $(M_n, n \in \mathbb{N})$ is a square-integrable martingale (i.e., $\mathbb{E}(M_n^2) < \infty$, $\forall n \in \mathbb{N}$), then the process $(M_n^2, n \in \mathbb{N})$ is a submartingale (as $x \mapsto x^2$ is convex).

2. **Stopping times**

**Definitions 2.1.** - A random time is a random variable $T$ with values in $\mathbb{N} \cup \{+\infty\}$. It is said to be finite if $T(\omega) < +\infty$ for every $\omega \in \Omega$ and bounded if there exists moreover an integer $N$ such that $T(\omega) \leq N$ for every $\omega \in \Omega$ (Notice that a finite random time is not necessarily bounded).

- Let $(X_n, n \in \mathbb{N})$ be a stochastic process and assume $T$ is finite. One then defines $X_T(\omega) = X_T(\omega) = \sum_{n \in \mathbb{N}} X_n(\omega) 1_{\{T=n\}}(\omega)$.

- A stopping time with respect to a filtration $(\mathcal{F}_n, n \in \mathbb{N})$ is a random time $T$ such that $\{T = n\} \in \mathcal{F}_n$, $\forall n \in \mathbb{N}$.

**Example.** Let $(X_n, n \in \mathbb{N})$ be a process adapted to $(\mathcal{F}_n, n \in \mathbb{N})$ and $a > 0$. Then $T_a = \inf\{n \in \mathbb{N} : |X_n| \geq a\}$ is a stopping time with respect to $(\mathcal{F}_n, n \in \mathbb{N})$. Indeed:

\[
\{T_n = n\} = \{|X_k| < a, \forall 0 \leq k \leq n-1 \text{ and } |X_n| \geq a\} = \bigcap_{k=0}^{n-1} \{X_k < a\} \cap \{|X_n| \geq a\} \in \mathcal{F}_n, \forall n \in \mathbb{N}.
\]
Definition 2.2. Let $T$ be a stopping time with respect to a filtration $(\mathcal{F}_n, n \in \mathbb{N})$. One defines the information one possesses at time $T$ as the following $\sigma$-field:

$$\mathcal{F}_T = \{ A \in \mathcal{F} : A \cap \{ T = n \} \in \mathcal{F}_n, \forall n \in \mathbb{N} \}.$$ 

Facts.
- If $T(\omega) = N \forall \omega \in \Omega$, then $\mathcal{F}_T = \mathcal{F}_N$. This is obvious from the definition.
- If $T_1$, $T_2$ are stopping times such that $T_1(\omega) \leq T_2(\omega) \forall \omega \in \Omega$, then $\mathcal{F}_{T_1} \subset \mathcal{F}_{T_2}$. Indeed, if $T_1(\omega) \leq T_2(\omega) \forall \omega \in \Omega$ and $A \in \mathcal{F}_{T_1}$, then for all $n \in \mathbb{N}$, we have:

$$A \cap \{ T = n \} = A \cap (\bigcup_{k=n}^{\infty} \{ T_1 = k \}) \cap \{ T_2 = n \} = \left( \bigcup_{k=n}^{\infty} A \cap \{ T_1 = k \} \right) \cap \{ T_2 = n \} \in \mathcal{F}_n,$$

so $A \in \mathcal{F}_{T_2}$. By the way, here is an example of stopping times $T_1$, $T_2$ such that $T_1(\omega) \leq T_2(\omega) \forall \omega \in \Omega$: let $0 < a < b$ and consider $T_1 = \inf \{ n \in \mathbb{N} : |X_n| \geq a \}$ and $T_2 = \inf \{ n \in \mathbb{N} : |X_n| \geq b \}$.
- A random variable $Y$ is $\mathcal{F}_T$-measurable if and only if $Y 1_{\{T=n\}}$ is $\mathcal{F}_n$-measurable, $\forall n \in \mathbb{N}$. As a consequence: if $(X_n, n \in \mathbb{N})$ is adapted to $(\mathcal{F}_n, n \in \mathbb{N})$, then $X_T$ is $\mathcal{F}_T$-measurable.

3 Doob's optional stopping theorem

Let $(M_n, n \in \mathbb{N})$ be a martingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ and $T_1$, $T_2$ be two stopping times such that $0 \leq T_1(\omega) \leq T_2(\omega) \leq N < \infty, \forall \omega \in \Omega$. Then

$$\mathbb{E}(M_{T_2} | \mathcal{F}_{T_1}) = M_{T_1} \text{ a.s.}$$

In particular, $\mathbb{E}(M_{T_2}) = \mathbb{E}(M_{T_1})$.

In particular, if $T$ is a stopping time such that $0 \leq T(\omega) \leq N < \infty, \forall \omega \in \Omega$, then

$$\mathbb{E}(M_T) = \mathbb{E}(M_0).$$

Remarks. - The above theorem says that the martingale property holds even if one is given the option to stop at any (bounded) stopping time.
- The theorem also holds for sub- and supermartingales (i.e., if $M$ is a submartingale, then $\mathbb{E}(M_{T_2} | \mathcal{F}_{T_1}) \geq M_{T_1} \text{ a.s.}$).

Proof. - We first show that if $T$ is a stopping time such that $0 \leq T(\omega) \leq N$, then $\mathbb{E}(M_N | \mathcal{F}_T) = M_T \text{ (*)}$: Indeed, let $Z = M_T = \sum_{n=0}^{N} M_n 1_{\{T=n\}}$. We check below that $Z$ is the conditional expectation of $M_N$ given $\mathcal{F}_T$:

(i) $Z$ is $\mathcal{F}_T$-measurable: $Z 1_{\{T=n\}} = M_n 1_{\{T=n\}}$, so $Z$ is $\mathcal{F}_T$-measurable.

(ii) $\mathbb{E}(ZU) = \mathbb{E}(M_N U), \forall U \in \mathcal{F}_T$-measurable and bounded:

$$\mathbb{E}(ZU) = \sum_{n=0}^{N} \mathbb{E}(M_n 1_{\{T=n\}} U) = \sum_{n=0}^{N} \mathbb{E}(\mathbb{E}(M_N | \mathcal{F}_n) 1_{\{T=n\}} U)_{\mathcal{F}_n \text{-measurable}} = \sum_{n=0}^{N} \mathbb{E}(M_N 1_{\{T=n\}} U) = \mathbb{E}(M_N U).$$

- Second, let us check that $\mathbb{E}(M_{T_2} | \mathcal{F}_{T_1}) = M_{T_1}$:

$$M_{T_1} \text{ (*) with } T=T_1 \quad \mathbb{E}(M_N | \mathcal{F}_{T_1}) \quad \mathbb{E}(\mathbb{E}(M_N | \mathcal{F}_{T_2}) | \mathcal{F}_{T_1}) \text{ (*) with } T=T_2 \quad \mathbb{E}(M_{T_2} | \mathcal{F}_{T_1}).$$

This concludes the proof of the theorem. \qed

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