Homework 3 (due Thursday, March 15)

Exercise 1. Let $\Omega = \{NW, NE, SW, SE\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ be the uniform distribution on $\Omega$. Let us also consider the 3 events:

$$A_1 = \{NW, NE\}, \quad A_2 = \{NW, SW\} \quad \text{and} \quad A_3 = \{NW, SE\}$$

a) Is the collection of events $\{A_1, A_2, A_3\}$ independent?

b) Are these events pairwise independent?

c) Assume now $\mathbb{P}(\{NW\}) = \mathbb{P}(\{NE\}) = 0.3$ and $\mathbb{P}(\{SW\}) = \mathbb{P}(\{SE\}) = 0.2$. Do the answers to the above two questions change?

Exercise 2. Let $X_1, X_2$ be independent and identically distributed (i.i.d.) random variables such that $\mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2$ for $i = 1, 2$. Let also $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

a) Are $Y$ and $Z$ independent?

b) Same question with $X_1, X_2$ i.i.d. $\sim \mathcal{N}(0, 1)$ random variables.

Exercise 3. Let $\Omega = \mathbb{R}^2$ and let us define on $\Omega$: $\mathcal{F} = \sigma(\{B_1 \times B_2, B_1, B_2 \in \mathcal{B}(\mathbb{R})\})$. Note that $\mathcal{F}$ is nothing but $\mathcal{B}(\mathbb{R}^2)$.

Let us also define the random variables $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally $\mu$ be a probability distribution on $\mathbb{R}$. Think e.g. of $\mu = \mathcal{N}(0, 1)$:

$$\mu(B) = \int_B \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad B \in \mathcal{B}(\mathbb{R}).$$

We are considering below two different probability measures defined on $(\Omega, \mathcal{F})$; we only specify them on the “rectangles” $B_1 \times B_2$ (Caratheodory’s extension theorem then guarantees that these probability measures can be extended uniquely to the whole $\sigma$-field $\mathcal{F}$ generated by the rectangles).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables $X_1$ and $X_2$.

Exercise 4. a) Let $X$ be a Poisson random variable with parameter $\lambda > 0$, i.e., $\mathbb{P}(\{X = k\}) = \frac{\lambda^k}{k!} e^{-\lambda}, k \geq 0$. Compute successively:

i) $\mathbb{E}(X)$

ii) $\mathbb{E}(X(X - 1))$

iii) $\text{Var}(X)$

b) Let $X$ be a centered Gaussian random variable of variance $\sigma^2$. Compute successively:

i) $\mathbb{E}(X^2)$

ii) $\mathbb{E}(X^4)$

iii) $\mathbb{E}(\exp(X))$

iv) $\mathbb{E}(\exp(-X^2))$

please turn the page %
Coding Exercise 5. Let $\mu \in \mathbb{R}$, $\sigma > 0$ and $X$ be a continuous random variable whose pdf is given by

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

a) Draw $M$ i.i.d. samples $X_1, \ldots, X_M$ according to $p_X$, compute numerically

$$\mu_M = \frac{1}{M} \sum_{j=1}^{M} X_j \quad \text{and} \quad \sigma_M = \sqrt{\frac{1}{M-1} \sum_{j=1}^{M} (X_j - \mu_M)^2}$$

and plot these two quantities as a function of $M$ on a graph. What do you observe as $M$ increases?

b) Plot the empirical cdf of $X_1, \ldots, X_M$ for three different values of $M$ (e.g., $M = 10$, $100$ and $1'000$). What do you observe as $M$ increases?

c) Consider now $M \times K$ i.i.d. samples $\{X_{m,k}, 1 \leq m \leq M, 1 \leq k \leq K\}$, with a fixed $K = 1'000$, as well as the empirical means

$$E_{M,k} = \frac{1}{M} (X_{1,k} + \ldots + X_{M,k}), \quad 1 \leq k \leq K$$

Plot the empirical cdf of $E_{M,1}, \ldots, E_{M,K}$ for the same three different values of $M$ as above. Again, what do you observe as $M$ increases?

Let again $\mu \in \mathbb{R}$, $\sigma > 0$ and $Y$ be a random variable whose pdf is given by

$$p_Y(y) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (y - \mu)^2}, \quad y \in \mathbb{R}$$

d-e-f) Same questions as in a-b-c) with the $X$ samples replaced by the $Y$ samples. (NB: In order to sample from $Y$, you should first compute its cdf).