Homework 3 (due Thursday, March 16)

Exercise 1. Check that the distributions below are well defined distributions and compute, when they exist, the mean and the variance of these distributions.

A) Discrete distributions:

a) Bernoulli \( B(p) \), \( p \in [0, 1] \): \( P(X = 1) = p, \ P(X = 0) = 1 - p \).

b) binomial \( Bi(n, p) \), \( n \geq 1, \ p \in [0, 1] \): \( P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \ 0 \leq k \leq n \).

c) Poisson \( P(\lambda) \), \( \lambda > 0 \): \( P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ k \geq 0 \).

B) Continuous distributions:

d) uniform \( U([a, b]) \), \( a < b \): \( p_X(x) = \frac{1}{b-a} 1_{[a,b]}(x), \ x \in \mathbb{R} \).

e) Gaussian \( N(\mu, \sigma^2) \), \( \mu \in \mathbb{R}, \ \sigma > 0 \): \( p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right), \ x \in \mathbb{R} \).

f) Cauchy \( C(\lambda) \), \( \lambda > 0 \): \( p_X(x) = \frac{1}{\pi} \frac{\lambda}{x^2 + \lambda^2}, \ x \in \mathbb{R} \).

Exercise 2. Let \( X \) be a centered Gaussian random variable of variance \( \sigma^2 \). Compute:

a) \( E(X^4) \).

b) \( E(\exp(X)) \).

c) \( E(\exp(-X^2)) \).

Exercise 3. (reverse Chebyshev’s inequality)

Let \( X \) be a square-integrable random variable such that \( X \geq 0 \) a.s. Let also \( 0 \leq t < E(X) \).

a) Show that

\[
P(\{X > t\}) \geq \frac{(E(X) - t)^2}{E(X^2)}.
\]

*Hint:* Use Cauchy-Schwarz’s inequality.

b) *Application:* Check that the above inequality holds in the particular case \( X \sim P(\lambda) \) and \( t = 0 \).
Exercise 4. Let $X$ be a centered random variable with variance $\sigma^2$. Using Chebyshev’s inequality, show that:

a) $\mathbb{P}(\{|X| \geq a\}) \leq \frac{\sigma^2}{a^2}$ and $\mathbb{P}(\{|X| \geq a\}) \leq \frac{2\sigma^2}{a^2 + \sigma^2}$.

b) $\mathbb{P}(\{X \geq a\}) \leq \frac{\sigma^2}{a^2 + \sigma^2}$ (use $\psi(x) = (x + b)^2$ with $b \geq 0$, then minimize over $b$).