Exercise 1. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk on \(\mathbb{Z}\) and \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration.

a) Is the process \((S^4_n, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

b) Is the process \((S^4_n - n, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

Hint: Recall that \((x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\).

c) Show that \(E(S^4_{n+1}) = E(S^4_n) + 6n + 1\) and deduce the value of \(E(S^4_n)\) by induction on \(n\).

Hint: Recall that \(\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\).

d) Compute \(\lim_{n \to \infty} \frac{E(S^4_n)}{n^2}\). Can you make a parallel with something you already know?

Exercise 2. Let \(Y = (Y_n, n \in \mathbb{N})\) be the process defined recursively as

\[
Y_0 = 1, \quad Y_{n+1} = \begin{cases} 
3Y_n \over 2, & \text{with probability } 1/2 \\
Y_n \over 2, & \text{with probability } 1/2 
\end{cases}
\]

a) Is the process \(Y\) a submartingale, supermartingale or martingale with respect to its natural filtration \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

b) Compute \(E(Y_n)\) and \(\text{Var}(Y_n)\) recursively, for all \(n \geq 1\).

c) Is the process \(Y\) confined to some interval?

d) Does there exist a random variable \(Y_\infty\) such that \(Y_n \to Y_\infty\) almost surely?

e) If it exists, what is the random variable \(Y_\infty\)?

Hint: In order to answer this question rigorously, consider the process \(Z\) defined as \(Z_n = \log(Y_n)\).

f) If \(Y_\infty\) exists, does it also hold that \(Y_n = E(Y_\infty | \mathcal{F}_n)\)?
Coding Exercise 3. Let \((M_n, n \in \mathbb{N})\) be a square-integrable martingale with respect to some filtration \((\mathcal{F}_n, n \in \mathbb{N})\) and \((H_n, n \in \mathbb{N})\) be a predictable process with respect to \((\mathcal{F}_n, n \in \mathbb{N})\), such that \(|H_n(\omega)| \leq K_n\) for every \(\omega \in \Omega\) and \(n \in \mathbb{N}\).

Let also \((G_n, n \in \mathbb{N})\) be the process defined as \(G_0 = 0, G_n = \sum_{j=1}^n H_j (M_j - M_{j-1}), n \geq 1\). By the proposition seen in class, we know that \(G\) is a martingale.

a) Show that \(\mathbb{E}(G_n^2) = \sum_{j=1}^n \mathbb{E}\left(\left(H_j^2 (A_j - A_{j-1})\right)\right), \) for every \(n \geq 1\), where \((A_n, n \in \mathbb{N})\) is the (unique) predictable and increasing process such that \((M_n^2 - A_n, n \in \mathbb{N})\) is a martingale.

b) Consider \(M = S\), the simple symmetric random walk. Find a sufficient condition on the process \(H\) (other than \(H \equiv 0 :\)) such that there exists a random variable \(G_\infty\) with \(\mathbb{E}(G_\infty | \mathcal{F}_n) = G_n\), for every \(n \in \mathbb{N}\).

c) Numerical application: still with \(M = S\) (i.e., \(M_n = S_n = \sum_{j=1}^n X_j\) with \(X_j\) i.i.d. \(\pm 1\) with equal probability), observe numerically how does the process \(G\) behave when \(n \to +\infty\) with the following \(H\)’s (which are all equal to 0 at time 0, by convention)

\[
H_n^{(1)} = \frac{1}{n} \quad H_n^{(2)} = \frac{X_{n-1}}{n} \quad H_n^{(3)} = \frac{X_{n-1}}{\sqrt{n}} \quad H_n^{(4)} = \frac{X_n}{\sqrt{n}} \quad H_n^{(5)} = \frac{\sum_{j=1}^{n-1} X_j}{n}
\]

NB: One of these \(H\)’s is problematic!