Exercise 1. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk on \(\mathbb{Z}\) and \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration.

a) Is the process \((S_n^4, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

b) Is the process \((S_n^4 - n, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

Hint: Recall that \((x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\).

c) Show that \(E(S_{n+1}^4) = E(S_n^4) + 6n + 1\) and deduce the value of \(E(S_n^4)\) by induction on \(n\).

Hint: Recall that \(\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\).

d) Compute \(\lim_{n \to \infty} \frac{E(S_n^4)}{n^2}\). Can you make a parallel with something you already know?

Exercise 2. Let \(Y = (Y_n, n \in \mathbb{N})\) be the process defined recursively as

\[
Y_0 = 1, \quad Y_{n+1} = \begin{cases} 
\frac{3Y_n}{2}, & \text{with probability 1/2,} \\
\frac{Y_n}{2}, & \text{with probability 1/2.}
\end{cases}
\]

a) Is the process \(Y\) a submartingale, supermartingale or martingale with respect to its natural filtration \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

b) Compute \(E(Y_n)\) and \(\text{Var}(Y_n)\) recursively, for all \(n \geq 1\).

c) Is the process \(Y\) confined to some interval?

d) Does there exist a random variable \(Y_\infty\) such that \(Y_n \xrightarrow{n \to \infty} Y_\infty\) almost surely?

e) If it exists, what is the random variable \(Y_\infty\)?

Hint: In order to answer this question rigorously, consider the process \(Z\) defined as \(Z_n = \log(Y_n)\).

f) If \(Y_\infty\) exists, does it also hold that \(Y_n = E(Y_\infty | \mathcal{F}_n)\)?