Exercise 1. Let \((\xi_n, n \geq 1)\) be a sequence of i.i.d. centered and bounded random variables and let \((\mathcal{F}_n, n \geq 1)\) be the filtration defined as \(\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \geq 1\). Among the following processes \((X_n, n \geq 1)\), which are martingales with respect to \((\mathcal{F}_n, n \geq 1)\)? (no justification needed here)

a) \(X_n = \xi_n, n \geq 1\).

b) \(X_1 = \xi_1, X_{n+1} = aX_n + \xi_{n+1}, n \geq 1 (a > 0 \text{ fixed})\).

c) \(X_1 = \xi_1, X_{n+1} = \xi_n + \xi_{n+1}, n \geq 1\).

d) \(X_n = \max(\xi_1, \ldots, \xi_n), n \geq 1\).

e) \(X_1 = \xi_1, X_n = \sum_{i=1}^{n}(\xi_1 + \ldots + \xi_{i-1})\xi_i, n \geq 1\).

Exercise 2. Let \((X_n, n \in \mathbb{N})\) be a submartingale and \(\varphi : \mathbb{R} \to \mathbb{R}\) be a Borel-measurable and convex function such that \(E(|\varphi(X_n)|) < \infty, \forall n \in \mathbb{N}\).

a) What additional property of \(\varphi\) ensures that the process \((\varphi(X_n), n \in \mathbb{N})\) is also a submartingale?

b) In particular, which of the following processes is ensured to be a submartingale: \((X_n^2, n \in \mathbb{N})\) or \((\exp(X_n), n \in \mathbb{N})\)?

Exercise 3. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk and \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration. Among the following random times, which are stopping times with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? which are bounded? (no formal justification needed here)

a) \(T = \sup\{n \geq 0 : S_n \geq a \text{ and } n \leq N\} (a > 0 \text{ and } N > 1 \text{ are fixed})\)

b) \(T = \inf\{n \geq 1 : S_n = \max_{0 \leq k \leq n} S_k\}\)

c) \(T = \inf\{n \geq 0 : S_n = \max_{0 \leq m \leq N} S_m\} (N \geq 1 \text{ is fixed})\)

d) \(T = \inf\{n \geq 0 : S_n \geq a \text{ or } n \geq N\} (a > 0 \text{ and } N \geq 1 \text{ are fixed})\)

Exercise 4. (“The” martingale)

A player bets on a sequence of i.i.d. (and balanced) coin tosses: at each turn, the player wins twice his bet if the coin falls on “heads” or loses his bet if the coin falls on “tails”.

Assume now that the player adopts the following strategy: he starts by betting 1 franc. If he wins his bet (that is, if the outcome is “heads”), he quits the game and does not bet anymore. If he loses (that is, if the outcome is “tails”), he plays again and doubles his bet for the next turn. He then goes on with the same strategy for the rest of the game.

We assume here that the player can borrow any money he wants in order to bet. Of course, we also assume that he has no information on the outcome of the next coin toss while betting on it.

a) Is the process of gains of the player a martingale (by convention, we set the gain of the player at time zero to be equal to zero)?

b) What is the gain of the player at the first time “heads” comes out?

c) Isn’t there a contradiction between a) and b)?