Exercise 1. Let $\Omega = \{1, \ldots, 6\}$ et $A = \{\{1,3,5\}, \{1,2,3\}\}$.

(a) Describe $F = \sigma(A)$, the $\sigma$-field generated by $A$.

Hint: For a finite set $\Omega$, the number of elements of a $\sigma$-field on $\Omega$ is always a power of 2.

(b) Give the list of non-empty elements $G$ of $F$ such that

if $F \in F$ and $F \subseteq G$, then $F = \emptyset$ or $G$.

These elements are called the atoms of the $\sigma$-field $F$ (cf. course). They form a partition of the set $\Omega$ and they also generate the $\sigma$-field $F$ in this case.

Exercise 2. Let now $\Omega = [0,1]$ and $F = B([0,1])$ be the Borel $\sigma$-field on $[0,1]$.

(a) What are the atoms of $F$?

(b) Is it true in this case that the $\sigma$-field $F$ is generated by its atoms?

Exercise 3. Let $\Omega = \{(i,j) : i,j \in \{1, \ldots, 6\}\}$, $F = \mathcal{P}(\Omega)$ and define the random variables $X_1(i,j) = i$ and $X_2(i,j) = j$.

(a) What are $\sigma(X_1)$, $\sigma(X_2)$?

(b) Is $X_1 + X_2$ measurable with respect to one of these two $\sigma$-fields?

Exercise 4. Let $F$ be a $\sigma$-field on a set $\Omega$ and $X_1, X_2$ be two $F$-measurable random variables taking a finite number of values in $\mathbb{R}$. Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subseteq \sigma(X_1, X_2)$, i.e., that $X_1, X_2$ carry together at least as much information as $Y$, but that the reciprocal statement is not necessarily true.

(a) Provide a non-trivial example of random variables $X_1, X_2$ such that $\sigma(Y) = \sigma(X_1, X_2)$.

(b) Provide a non-trivial example of random variables $X_1, X_2$ such that $\sigma(Y) \neq \sigma(X_1, X_2)$.

(c) Assume that $X_1$ and $X_2$ share at least two different values $a \neq b \in \mathbb{R}$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?

Exercise 5. Let $\Omega = [-1,1]$ and $(X_i, i = 1, \ldots, 4)$ be a family of random variables on $\Omega$ defined as

$$X_i(\omega) = \begin{cases} 
1 & \text{if } \frac{i-1}{4} < \omega \leq \frac{i}{4}, \\
(-1)^i & \text{if } -\frac{i}{4} < \omega \leq -\frac{i-1}{4}, \\
0 & \text{otherwise}.
\end{cases}$$

Describe the $\sigma$-field $F = \sigma(X_i, i = 1, \ldots, 4)$ using its atoms.